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#

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**Hardy, Littlewood, Ramanujan and Cartwright**

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**Slide: Title**

Thank you for coming to my lecture today.

You will see that I have amended the title to include Mary Cartwright, because this talk is about the value of collaboration between mathematicians and I wanted to include the work between Littlewood and Cartwright during the Second World War.

**Slide: Hardy, Littlewood and Ramanujan**

The lengthy and fruitful collaboration of G. H. Hardy and J. E. Littlewood was the most productive in mathematical history. Dominating the English mathematical scene for the first half of the 20th century, they produced a hundred joint papers of great influence, most notably in analysis and number theory. Into their world came Ramanujan, one of the most brilliant and intuitive mathematicians of all time, who left India to work with them in Cambridge until his untimely death at the age of 32.

**Slide: Hardy**

Hardy was born in Cranleigh, in Surrey. His parents were schoolteachers and he had an enlightened upbringing in a typical religious Victorian household, although in adult life he was a militant atheist. He attended Winchester College before proceeding to Trinity College, Cambridge in 1896.

Hardy’s first research paper in 1900 was on integration; he later wrote another sixty-eight papers on the same subject.

**Slide: Title page of *A Course of Pure Mathematics***

His textbook, *A Course of Pure Mathematics,* was published in 1908. This book had a tremendous influence on generations of mathematicians in the British Isles. It was a model of clarity and presented elementary analysis to students in a rigorous yet accessible way. Previously analysis had been a neglected area of mathematics even in Cambridge. Littlewood later described the book’s author as *a missionary talking to cannibals.*

This is the title page of the third edition published in 1921 and available online at: <https://www.gutenberg.org/files/38769/38769-pdf.pdf>

In the preface to the third edition Hardy wrote:

*It is curious to note how the character of the criticisms I have had to meet has changed. I was too meticulous and pedantic for my pupils of fifteen years ago: I am altogether too popular for the Trinity scholar of to-day. I need hardly say that I find such criticisms very gratifying, as the best evidence that the book has to some extent fulfilled the purpose with which it was written.*

Also in that year, 1908, and in spite of his lifelong distaste for applied mathematics, he made a significant contribution to a problem in genetics, using only simple algebra and probability, and sent it to the journal *Science*. It is now known as the Hardy-Weinberg Law.

**Slide: Hardy-Weinberg Law**

The complete article is on the left and is, as you can see, short. The title of the article is *Mendelian Proportions in a Mixed Population*. In 1908, of course, the mathematical and biological principles of genetics were in their infancy. Let me describe what he did. in genes of type uppercase *A* or lowercase *a,* where *A* is dominant and *a* is recessive. So if an offspring receives an *A* gene from **either** parent then the offspring will have the characteristic, say brown eyes, but if the offspring receives an *a* from **both** parents then the child will have the opposite characteristic, say blue eyes. Before Hardy’s work it was thought that a dominant characteristic would over generations become so widespread as to eliminate its recessive counterpart. Hardy showed in this article in *Science* that:

*In a word, there is not the slightest foundation for the idea that a dominant character should show a tendency to spread over a whole population, or that a recessive should tend to die out.*

Let me show you why this follows. I will first set up some notation to describe the situation.

**Slide: Setting things up**

Now each parent contributes one gene so the child’s genetic make-up must be one of the three types: *AA* or *Aa* or *aa.*

Suppose the probability of having an *AA* pair is $ \frac{p}{n}$, an *Aa* pair $\frac{2q}{n}$, and an *aa* pair $\frac{r}{n}$. These three probabilities add up to 1.

Also assume, as did Hardy that these probabilities are the same among males and females and also that mating is random.

I shall now calculate the probability that a parent contributes an uppercase *A* gene to its child.

**Slide: What is the probability that a parent contributes an *A* gene to its child?**

Then we ask the probability that a parent contributes an *A* gene to its child. The probability is 1 if the parent is AA, a ½ if the parent is Aa and 0 if the parent is aa. So

Probability (parent contributes an *A* gene) = 1.$ \frac{p}{n}$ + $\frac{1}{2}\frac{2q}{n}$ + 0.$\frac{r}{n}$ = $\frac{p+q}{n}$

Probability (parent contributes an *a* gene) = 0.$ \frac{p}{n}$ + $\frac{1}{2}\frac{2q}{n}$ + 1.$\frac{r}{n}$ = $\frac{q+r}{n}$

The next thing is to calculate the probabilities of the three genetic types in the *next* generation.

**Slide: What now are the probabilities of the three genetic types in the *next* generation?**

Probability (*AA*) = Probability (male parent gives *A* and female parent gives *A*)

= Probability (male parent gives *A*)x Probability ( female parent gives *A*)

= $\frac{p+q}{n} . \frac{p+q}{n}$ = $\frac{(p+q)^{2}}{n^{2}}$. Because of the assumption of random mating we can use independence to multiply the probabilities.

Probability (*Aa*) = Probability (male parent gives *A* and female parent gives *a*)

 + Probability (male parent gives *a* and female parent gives *A*)

= $\frac{p+q}{n} . \frac{q+r}{n}$ + $\frac{q+r}{n} . \frac{p+q}{n}$ = $\frac{2(p+q)(q+r)}{n^{2}}$

Probability (*aa*) = Probability (male parent gives *a* and female parent gives *a*)

= $\frac{q+r}{n} . \frac{q+r}{n}$ = $\frac{(q+r)^{2}}{n^{2}}$

Let me put the results so far into a table.

**Slide: The probabilities of the different genetic types in the second generation**

These probabilities of the different genetic types in the second generation are **not** necessarily the same as the probabilities of the different genetic types in the first generation. We have:

|  |  |  |  |
| --- | --- | --- | --- |
|  | *AA* | Aa | aa |
| First generation | $$\frac{p}{n}$$ | $$\frac{2q}{n}$$ | $$\frac{r}{n}$$ |
| Second generation | $$\frac{(p+q)^{2}}{n^{2}}$$ | $$\frac{2(p+q)(q+r)}{n^{2}}$$ | $$\frac{(q+r)^{2}}{n^{2}}$$ |

**But** if we do the calculations for the third generation they are the same as for the second. It is the same algebra going from the second to the third generation as we did going from the first to the second generation.

**Slide: After the second generation**

|  |  |  |  |
| --- | --- | --- | --- |
|  | *AA* | Aa | aa |
| First generation | $$\frac{p}{n}$$ | $$\frac{2q}{n}$$ | $$\frac{r}{n}$$ |
| Second generation | $$\frac{(p+q)^{2}}{n^{2}}$$ | $$\frac{2(p+q)(q+r)}{n^{2}}$$ | $$\frac{(q+r)^{2}}{n^{2}}$$ |
| Third generation | $$\frac{(p+q)^{2}}{n^{2}}$$ | $$\frac{2(p+q)(q+r)}{n^{2}}$$ | $$\frac{(q+r)^{2}}{n^{2}}$$ |

As Hardy put it:

*“…whatever the value of p, q and r may be, the distribution will in any case continue unchanged after the second generation.”*

**Slide: Exercise**

I leave you with an exercise:

Show that if in the first generation the proportions of the gene types are 3 : 6 :1 then in the second generation the proportions will be 9 : 12 : 4 and that this will remain the case in the third and all later generations.

**Slide: The Book of Presidents**

This short paper on genetics was published in 2008. Two years later Hardy was elected a Fellow of the Royal Society and subsequently received many of their most distinguished awards. Hardy had an unparalleled influence over British mathematics during the first half of the twentieth century. He was the only person to be twice President of the London Mathematical Society and he left copyright in his work to the LMS as well as a substantial portfolio of investments which enabled the LMS to broaden and expand its activities. It was the most generous benefaction the LMS has received. The London Mathematical Society was very close to his heart and he wrote that it had:

always meant much more to me than any other scientific society to which I have belonged.

At the end of his second term Hardy was succeeded as President by Littlewood. Let me tell you something about J.E. Littlewood.

**Slide: Littlewood**

Littlewood was eight years younger than Hardy and was born in Rochester, in Kent. He spent part of his youth in South Africa returning to England in 1900. He won a scholarship to Trinity College in 1903.

His first research paper was on integral functions, in 1906. His first job was as a Lecturer at Manchester University where he spent three years before returning to Trinity in 1910 where he lived for the rest of his life.

He did not enjoy his time at Manchester, to put it mildly, and later wrote about it:

*One day in a Manchester term, leaving my lodgings for the usual grim day’s work, I felt an unusual sense of well-being. I presently traced this to the fact that it was not actually raining.*

**Slide: Extension of Tauber’s theorem**

In 1910 he proved a deep result on the convergence of certain series. On the left we see part of the first page of the manuscript of the paper which says:



In his description of how he obtained the theorem he wrote:

*One day I was playing round with this, and a ghost of an idea entered my mind of making r, the number of differentiations, large. At that moment the spring cleaning that was in progress reached the room I was working in, and there was nothing for it but to go walking for 2 hours, in pouring rain*.

He continued:

*The problem seethed violently in my mind: the material was disordered and cluttered up with irrelevant complications cleared away in the final version, and the ‘idea' was vague and elusive. Finally I stopped, in the rain, gazing blankly for minutes on end over a little bridge into a stream (near Kenwith wood), and presently a flooding certainty came into my mind that the thing was done. The 40 minutes before I got back and could verify were none the less tense.*

**Slide: preparation, incubation, illumination and verification**

Later on in life in an article called *The Mathematician's Art of Work* he distinguished four phases in creative work: preparation, incubation, illumination and verification or working out. He viewed the last phase of verification as within the range of any competent mathematician given the illumination!

Preparation was largely conscious or anyhow directed by the conscious and consisted of stripping the problem to its essentials, surveying all relevant knowledge and considering possible analogues. Following Newton he advised that the problem should be kept constantly in mind during other periods of work.

Incubation is the work of the subconscious during the waiting time which may be several years.

He says that Illumination, which can happen in a fraction of a second, is the emergence of the creative idea into the consciousness and implies some mysterious rapport between the subconscious and the conscious. He recommends walking and the *relaxed activity of shaving* as helpful to the process of illumination.

**Slide: Collaboration with Hardy**

Returning to his description of how he obtained the convergence result we see at the top of the slide, he finished by writing:

*On looking back this time seems to me to mark my arrival at a reasonably assured judgment and taste, the end of my ‘education’. I soon began my 35-year collaboration with Hardy.*

It is to this remarkable collaboration that I now turn.

**Slide: Hardy and Littlewood**

The collaboration began in 1912. Both were geniuses, but Littlewood was probably the more original and imaginative while Hardy was the consummate craftsman, a master of stylish writing. In a 1947 lecture, their admirer, the eminent Danish mathematician Harald Bohr, brother of the Nobel Prize winning physicist Niels Bohr, reported a colleague as saying:

**Slide: Hardy, Littlewood, and Hardy–Littlewood**

*Nowadays, there are only three really great English mathematicians, Hardy, Littlewood, and Hardy–Littlewood.*

There is a story that at a conference Littlewood met a German mathematician who said he was most interested to discover that Littlewood really existed, as he had always assumed that Littlewood was a name used by Hardy for lesser work which he did not want to put out under his own name: Littlewood apparently roared with laughter.

Harald Bohr also tells how Hardy and Littlewood planned their co-operation in such a way as to preserve their individual personal freedom which was so important to them. As a safety measure they formulated some axioms for their collaboration.

**Slide: Axiom 1 of Collaboration**

There were four axioms and the first was that when one wrote to the other it did not matter whether what they wrote was right or wrong. This was to allow them to write as they pleased. Incidentally they preferred to exchange ideas in writing rather than orally.

**Slide: Axiom 2 of Collaboration**

The second axiom was that when one wrote to the other the recipient was not only under no obligation to answer it but indeed under no obligation even to open it! They reasoned that the recipient might not want to work at that time or was then working on something else. I wonder what they would think of our world of email!

**Slide: Axiom 3 of Collaboration**

The third axiom seems to be an efficiency one. It was that although it did not matter if both of them worked on the same detail it was better if they didn’t.

**Slide: Axiom 4 of Collaboration**

The final fourth axiom might have been the most important. It said that it did not matter if one of them had not contributed the least bit to the contents of a paper appearing under their joint names. This was to avoid quarrels and difficulties about whose name would go on the paper.

As Harald Bohr remarked:

*I think one may safely say that seldom – or never – was such an important and harmonious collaboration founded on such apparently negative axioms.*

I want to tell you about one of the problems on which they collaborated and the problem is on the zeros of the Riemann zeta function.

This function lies at the heart of one of the most intriguing and difficult questions in mathematics: The Riemann Hypothesis.

**Slide: Riemann Hypothesis**

The question is:

*Do all the solutions of a certain equation have a particular form?*

The full statement is:

*Do all the non-trivial zeros of the Riemann Zeta function have real part 1/2?*

Answering this question could win you a million dollars!

**Slide: Riemann Hypothesis from Clay site**

The million dollar prize is offered by the Clay Mathematics Institute, now based in Oxford, and the solution to the Riemann Hypothesis is one of seven problems that were originally posed. One of these seven problems, the Poincaré conjecture, has now been solved.

The Clay Mathematics Institute proposed the prizes because as they say on their website:

*The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.*

*http://www.claymath.org/millennium-problems/riemann-hypothesis*

If you want to find out more the web address is on the hand-out, available at the end of the lecture.

Now to define the Riemann zeta function.

**Slide: Riemann Zeta function**

Its value at k is the sum of the reciprocals of the kth power of all the integers. When k = 1 we get:

**Slide: Riemann Zeta function with k = 1**

When k = 2 we get:

**Slide: Riemann Zeta function with k = 2**

Now crucially to importance of the zeta function is that it can be represented in terms of the primes. Let me remind you first of the Fundamental Theorem of Arithmetic.

**Slide: Fundamental Theorem of Arithmetic**

Every whole number can be written as a product of prime numbers in only one way apart from the order in which they are written. Primes are integers which are divisible only by one and themselves.

42 = 2 x 3 x 7

80 = 2 x 2 x 2 x 2 x 5

22012013 = 19 x 53 x 21859

Then let me illustrate the connection with the primes by showing how ζ(2) can be written as a product of series only involving primes.

**Slide: ζ(2) as a product**

**ζ(2) = 1 +** $\frac{1}{2^{2}}$ **+** $\frac{1}{3^{2}}$ **+** $\frac{1}{4^{2}}$ **+**$ \frac{1}{5^{2}}$ **+ ··· +** $\frac{1}{42^{2}}$ **+ ··· +** $\frac{1}{80^{2}}$ **+ ···**

**= (1 +** $\frac{1}{2^{2}}$ **+** $\frac{1}{2^{4}}$ **+** $\frac{1}{2^{6}}$ **+** $\frac{1}{2^{8}}$ **+** $\frac{1}{2^{10}}$ **+** $\frac{1}{2^{12}}$ **+** $\frac{1}{2^{14}}$ **+** $\frac{1}{2^{16}}$ **+··· )**

**× (1 +** $\frac{1}{3^{2}}$ **+** $\frac{1}{3^{4}}$ **+** $\frac{1}{3^{6}}$ **+** $\frac{1}{3^{8}}$ **+** $\frac{1}{3^{10}}$ **+** $\frac{1}{3^{12}}$ **+** $\frac{1}{3^{14}}$ **+** $\frac{1}{3^{16}}$ **+··· )**

**× (1 +** $\frac{1}{5^{2}}$ **+** $\frac{1}{5^{4}}$ **+** $\frac{1}{5^{6}}$ **+** $\frac{1}{5^{8}}$ **+** $\frac{1}{5^{10}}$ **+** $\frac{1}{5^{12 }}$ **+** $\frac{1}{5^{14}}$ **+** $\frac{1}{5^{16}}$ **+ ··· )**

**× (1 +** $\frac{1}{7^{2}}$ **+** $\frac{1}{7^{4}}$ **+** $\frac{1}{7^{6}}$ **+** $\frac{1}{7^{8}}$ **+** $\frac{1}{7^{10}}$ **+** $\frac{1}{7^{12}}$ **+** $\frac{1}{7^{14}}$ **+** $\frac{1}{7^{16}}$ **+··· )**

**× (1 +** $\frac{1}{11^{2}}$ **+** $\frac{1}{11^{4}}$ **+** $\frac{1}{11^{6}}$ **+** $\frac{1}{11^{8}}$ **+** $\frac{1}{11^{10}}$ **+** $\frac{1}{11^{12}}$ **+** $\frac{1}{11^{14}}$ **+** $\frac{1}{11^{16}}$ **+··· )**

**× ····**

Note how $\frac{1}{42^{2}}$ appears in the product because 422 = 22 x 32 x 72

**Slide: Show how** $\frac{1}{80^{2}}$ **appears in the product because 802 = 28 x 52**

Each of these series of primes is of a simple type called a geometric progression and can be summed so we get a formula for ζ(2) in terms of the primes. Indeed the same is true for ζ(k) for all integers k.

Now each of these series in the product is a geometric series – each term in the series is obtained from the previous one by multiplying by the same number, the common ratio. In the first it is$ \frac{1}{2^{2}}$, in the next $\frac{1}{3^{2}}$ then $\frac{1}{5^{2}}$ and so on. It is straightforward to write the sum of such a geometric progression it is just 1 over (1 minus the common ratio)

**Slide: Formula for ζ(s) in terms of product of primes**

Here we have the formula for ζ(s) in terms of product of expressions involving primes.

ζ(s) = 1 + $\frac{1}{2^{s}}$ + $\frac{1}{3^{s}}$ + $\frac{1}{4^{s}} + \frac{1}{5^{s}} + \frac{1}{6^{s}} + \frac{1}{7^{s}} + \frac{1}{8^{s}} + ···$

= $\prod\_{}^{}(\frac{1}{1- p^{-s}})$

Where the symbol $\prod\_{}^{}$ is the product over all prime values of p and

$\frac{1}{1- p^{-s}}$ = 1 + $\frac{1}{p^{s}}$ + $\frac{1}{p^{2s}}$ + $\frac{1}{p^{3s}}$ + $\frac{1}{p^{4s}}$ + $\frac{1}{p^{5s}}$ + $\frac{1}{p^{6s}}$ + $\frac{1}{p^{7s}}$ +···

So we now have this function, the Riemann Zeta function, defined for positive integers and whose value at each integer is closely related to the primes.

**Slide: ζ(s) defined for all real and complex s except for s = 1**

Riemann showed that the definition of the Zeta function can be extended not only to other real numbers but to almost all of the numbers in the complex plane. The only point in the complex plane where the Riemann Zeta function is not defined is at 1 because the value at 1 is the harmonic series, the sum of the reciprocals of the integers and this is infinite.

**Slide: Riemann Hypothesis - All non-trivial zeros lie on the line x = 1/2**

Riemann’s zeta function was defined in his only paper on number theory published in 1859 and in it his great contribution was to obtain an exact formula for the number of primes up to any particular value, x, and this formula involved in a crucial way the zeros of the Zeta function.

A zero of the Zeta function is a value of *s* such that ζ(*s*) = 0.

There are zeros of the Zeta function at the numbers -2, -4, -6, ···, which are called the trivial zeros and all the other zeros lie within a vertical strip between 0 and 1. This is called the critical strip.

The first few zeros in the critical strip are:

$^{1}/\_{2}$ ± 14.1 *i,* $ ^{1}/\_{2}$ ± 21.02 *i,* $^{1}/\_{2}$ ± 25.01 *i ,* $^{1}/\_{2}$ ± 30.4 *i, ···*

But these all have the form ½ plus something times i. The Riemann hypothesis is that all the zeros in the critical strip have this form. In other words the hypothesis is that they all lie on the central vertical line.

Hardy was one of the first mathematicians to do important work on the Riemann hypothesis and in 1914 he showed that there were infinitely many zeros on the central critical line and that in a certain sense the zeta function was ‘small’ on the critical line. He was not able to prove that all the non-trivial zeros lay on the critical line but it was a major advance. Later with Littlewood he proved deep extensions of these results.

Let me now bring Ramanujan into the story.

**Slide: Srinivasa Ramanujan 1887–1920**

In 1913, Bertrand Russell wrote, in one of his over 2,000 letters to his lover, Lady Ottoline Morrell:

*In Hall I found Hardy and Littlewood in a state of wild excitement, because they believe they have found a second Newton …*

*Hardy has written to the Indian Office and hopes to get the man here at once.*

This second Newton was Srinivasa Ramanujan, who had written to Hardy submitting his mathematical discoveries on prime numbers, series and integrals.

**Slide: Ramanujan’s first letter to Hardy**

On the slide on the left we have the start of Ramanujan’s letter:

*Dear Sir,*

*I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course followed in a University course, but I am striking out a new path for myself. I have made special investigation of divergent series in general and the results I get are termed by the local mathematicians as startling.*

The right hand side shows a facsimile of a page of the letter showing some of his results on continued fractions.

Although some of his results were incorrect, others showed amazing insight, and Hardy and Littlewood surmised that they must be correct since no-one would have the imagination to make them up. Ramanujan was clearly a genius of the first order, but untutored in formal mathematics.

Hardy and Littlewood invited him to Cambridge, and he arrived in April, 1914, where they collaborated on several ground-breaking papers.

During the First World War Littlewood was a Second Lieutenant in the Royal Garrison Artillery and worked on ballistic problems as well as continuing his collaborations.

Hardy, an ardent pacifist remained in Cambridge during the First World War, collaborating with Ramanujan. One of their most fruitful areas of work was on partitions.

**Slide: Partitions**

The problem is to find the number of different ways, *p(n),* of writing a given positive integer, *n*, as the sum of integers.

**Example: *p(4) = 5***

Because 4 has the five partitions *4, 3 + 1, 2 + 2, 2 + 1 + 1,1 + 1 + 1 + 1*

Note that in the partitions, order does not matter so that 3 + 1 is the same as 1 + 3.

**Slide: Example: *p(7) = 15***

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 7 | 6 + 1 | 5 + 2 | 5 + 1 + 1 | 4 + 3 |
| 4 + 2 + 1 | 4 + 1 + 1 + 1 | 3 + 2 + 2 | 3 + 3 + 1 |  |
| 3 + 2 + 1 + 1 | 3 + 1 + 1 + 1 + 1 | 2 + 2 + 2 + 1 |  |  |
| 2 + 2 + 1 + 1 + 1 | 2 + 1 + 1 + 1 + 1+ 1 |  |  |  |
| 1 + 1 + 1 + 1 + 1 + 1+ 1 |  |  |  |  |

By *n = 70* the number of partitions has risen to over 4 million and

*p(200) = 3,972,999,029,388* which is nearly four thousand billion.

Hardy and Ramanujan were able to find an exact formula for *p(n)* for any value of *n,* a result which Littlewood described as very astounding. The formula involved 24th complex roots of unity, exponentials, derivatives and summations. Littlewood felt that this result brought out the best and most characteristic work from the two of them.

**Slide: Theorem of the day**

Further information about the partition function and its rate of growth can be found at Robin Whitty’s website *Theorem of the Day* which is an excellent resource of various theorems.

http://www.theoremoftheday.org/NumberTheory/Partitions/TotDPartitions.pdf

On this slide we see an approximation which is derived from their exact solution:

$$p\left(n\right)\~ \frac{1}{4n\sqrt{3}} e^{2π\sqrt{n/6}}$$

In the middle are graphs which plots *p(n)* and the approximation. On the right at the bottom is a graph plotting their ratio which tends to 1 as n gets large.

But in 1917, as a result of the climate and a poor diet, Ramanujan fell ill and was diagnosed with tuberculosis.

**Slide: Taxicab number story**

A well-known story tells of Hardy visiting him in hospital:

*I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen.*

*“No,” he replied, “it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.”*

*1729 = 13 + 123 = 93 + 103*

**Slide: Taxicab picture and taxicab number**

On the right is London Taxi cab n°**1729**. Built in France by Unic, they were the most current taxi cabs in London at the time of the Hardy/Ramanujan anecdote.

The Image is from the "London Vintage Taxi Association" but slightly modified: LF 5795 becomes here LF 1729! I found it at © <http://www.christianboyer.com/taxicab/> where we have an investigation of taxicab numbers.

*Taxicab(n)* is the smallest number expressible as a sum of two positive cubes in n different ways. So *Taxicab(2) = 1729 and Taxicab(3) = 87539319 [ = 1673 + 4363 = 2283 + 4233 = 2553 + 4143 ]*

The search is not fruitless because Fermat had proved that numbers expressible as a sum of two cubes in n different ways exist for any n.

Hardy and Littlewood arranged for Ramanujan to be elected as a Fellow of the Royal Society and as a Fellow of Trinity College. Ramanujan was only the second Indian to be elected FRS and he was delighted by the honour and recognition. He was however still ill and he returned to India in 1919, and died the following year.

**Slide: Cover of *The Man Who Knew Infinity***

On this slide we see a fine memorial bust of Ramanujan at a museum in Calcutta. On the right is the cover of his excellent biography by Robert Kanigel.

Hardy was devastated:

“*For my part, it is difficult for me to say what I owe to Ramanujan – his originality has been a constant source of suggestion to me ever since I knew him, and his death is one of the worst blows I have ever had.”*

Hardy worked through Ramanujan’s papers and notebooks and he was one of the editors of his collected works.

By 1919, when Ramanujan had returned to India, Hardy felt the need for a break from Cambridge. The war years had been very difficult for him as a pacifist at Trinity College and he was sickened by his colleague’s views and in particular by their treatment of his friend Bertrand Russell.

**Slide: The Oxford years**

He was appointed Savilian Professor of Geometry in Oxford, where he spent eleven years. This is the first of a series of Posters created by a team of people including Robin Wilson and myself and freely available at the Oxford Mathematical Institute website:

**https://www.maths.ox.ac.uk/system/files/attachments/GH%20Hardy.pdf**

Of this period, Hardy claimed that:

*I was at my best at a little past forty, when I was a professor at Oxford.*

C.P. Snow wrote that this was the happiest time of his life and Littlewood wrote that:

*He preferred the Oxford atmosphere and said that they took him seriously, unlike Cambridge.*

**Slide: Hardy leading out the cricket team**

Hardy was fanatical about cricket both as a spectator and a player. Here we see him leading out a cricket team of Oxford mathematicians during a British Association meeting in 1926: Hardy dubbed this photograph *Mathematicians v. The Rest of the World.*

At Oxford he reformed the mathematics curriculum and built up an impressive school of analysis.

**Slide: Hardy at New College and manuscript page**

Before Hardy there was no flourishing research tradition in Oxford. Hardy determined to change all this. His own research interests were mainly in number theory and mathematical analysis, and his greatest legacy to Oxford mathematics was the internationally renowned research school of analysis that he established there in the 1920s.

His research output blossomed: during his Oxford years he wrote a hundred papers, over half of them with Littlewood who was still in Cambridge.

One of the problems on which Hardy and Littlewood collaborated was due to Edward Waring, an eighteenth century English mathematician.

**Slide: Waring’s problem**

Waring asserted on the basis of empirical evidence that:

*Every positive integer can be written as the sum of at most 4 squares, 9 cubes or 19 fourth powers.*

And asked

*Does this continue for nth powers, and if so how many powers are needed?*

**Slide: Hilbert’s answer**

The great German mathematician David Hilbert showed in 1909, using complicated algebraic identities, that the answer was yes. Every positive integer can be written as the sum of fifth powers, sixth powers, seventh powers and so on. But the information he could obtain on *how many* of the powers were needed was weak.

Hardy and Littlewood, in a series of ground-breaking papers, introduced a method, the *circle* method, to help determine how many powers were needed. This circle method had originated in the work Hardy and Ramanujan had done on partition functions but its use by Hardy and Littlewood in Waring’s problem was more difficult and technical. They used the method to prove that Waring was right about fourth powers, at least for sufficiently large integers. They proved that that every sufficiently large number was the sum of at most 19 fourth powers. It was only in 1986 that it was proved for all integers.

The circle method also provides a possible approach to deciding on Goldbach’s conjecture which is that every even number greater than two is the sum of two primes. The circle method is still of current interest.

**Slide: Terence Tao**

On this slide I show part of the personal mathematics blog of Terence Tao who was a co-recipient of the 2006 Fields Medal and the 2014 Breakthrough Prize in Mathematics. In this article he discusses the relevance of the circle method to various classical problems in number theory. Tao’s blog has been described by another Fields medal winner, Timothy Gowers, a Fellow of Trinity College, Cambridge, as *the undisputed king of all mathematics blogs.*

In 1931 the Sadleirian Chair in Cambridge fell vacant. Hardy decided to apply, apparently for two reasons. The first was that in spite of his efforts at Oxford Cambridge was still the mathematical centre of England. The second was that he was thinking of his retirement. At Trinity in Cambridge, unlike New College in Oxford, he could stay in his rooms after he retired. Hardy was duly appointed, and spent the rest of his life back at Trinity.

**Slide: Mathematicians Apology and Mathematical Miscellany**

Both Hardy and Littlewood wrote well known books explaining the nature of mathematics to a general readership and both of them are available free online at archive.org at the URLs:

<https://archive.org/details/AMathematiciansApology>

https://archive.org/details/mathematiciansmi033496mbp

Hardy’s *A Mathematician’s Apology* (1940) is a personal and rather melancholy account by a mathematician looking back as his powers are waning, while Littlewood’s *A Mathematician’s Miscellany* (1953) is a more joyful work, full of mathematical gems, that allows his readers to experience academic life at Trinity through his perceptive eyes.

Hardy died on the very day that he was due to be presented with the Copley Medal by the Royal Society. His epitaph could well be this sentence from *A Mathematican’s Apology:*

*I still say to myself when I am depressed, and find myself forced to listen to pompous and tiresome people, “Well, I have done one thing you could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms.*

Littlewood outlived him by thirty years and I want to finish with a brief description of Littlewood’s collaboration with Mary Cartwright.

**Slide: Mary Cartwright**

Mary Cartwright was born in 1900 and went to Oxford University, receiving first class honours in mathematics – women had only a couple of years earlier been allowed to sit the final examination. She was then a school teacher for a while but returned in 1928 to Oxford and was one of Hardy’s group of students. Littlewood was the external examiner for her doctoral examination and recalls in his Mathematical Miscellany:

*I first met Miss Cartwright as an Oxford Ph.D. candidate. At the oral my co-examiner asked a question so silly and unreal that she was completely taken aback, and blushed. I was able to get in a wink, and I think it restored her nerve.*

She then moved to Girton College Cambridge as a Research Fellow. In January 1938 the radio research board issued an appeal for help in solving certain systems of non-linear differential equations which arose in modelling the behaviour of thermionic valves. Presumably the request was connected with potential military applications, in particular Radio Detection and Ranging - what was soon to become known as radar.

Littlewood was a formidable applied mathematician – he worked on ballistics, as I’ve said during the First World War - and he collaborated with Cartwright on this problem, known as the van der Pol oscillator.. Littlewood and Cartwright were the first to combine analytical and topological methods to discover phenomena that later would be called *chaotic* showing that *chaos* could arise in real engineering problems.

**Slide: Cartwight and Littlewood title page from 1945**



Non-linear differential equations are usually difficult to solve because it is not possible as in linear equations to build up the solution to complicated problems from simpler ones.

The Cambridge mathematician Tom Körner says of their work:

*The work of Cartwright, Littlewood and their American counterparts changed the way we think about differential equations for ever. They changed the questions we ask and the answers we expect. In emphasising the topological aspect of the subject they both joined and accelerated a swing now visible over much of mathematics towards a ‘regeometricisation’ of the way we think of things.*

**Slide: Lisa Jardine on *A Point of View***

You can find out more about Mary Cartwright’s work on radar from Lisa Jardine’s *Point of View* radio discussion which is at: <http://www.bbc.co.uk/news/magazine-21713163>

On International Women's Day 2013, Lisa Jardine, presenting *A Point of View,* felt it was a particularly appropriate time to blow Dame Mary Cartwright's trumpet on her behalf - for her brilliance as a mathematician, and as one of the founders of the important field of chaos theory.

Let me finish my discussion of Hardy, Littlewood, Ramanujan and Cartwright and their mathematical work by reading the last paragraph of Hardy’s *A Mathematician’s Apology*.

*The case for my life then, or for that of anyone else who has been a mathematician in the same sense in which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.*

Thank you.

Next time I will be speaking about John Von Neumann and Alan Turing.

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