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THE RIDDLE OF THE VANISHING CAMEL: FROM PUZZLES TO NUMBER THEORY

A Lecture by

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Gresham Lecture

The Riddle of the Vanishing Camel

Ian Stewart

A lost tale from the Arabian Nights...

"... and now, my dear," said the Sultan, "it is time for you to continue with your story. After which, to my regret, I shall be forced to replace you, as is my custom..."

"As your Lordship commands," replied Scheherezade. But secretly she smiled, for her plan had postponed her demise these past several months, and still Shahryar failed to understand that he was being led by his regal nose. "You recall that I was telling of the Bedouin patriarch Mustapha ibn Mokhta, who after his terrible encounter with the giant snake from the bottomless oasis lay close to dying..."

The cluster of black goat-hair tents shimmered in the heat among the vast sand dunes of the northern Arabian desert. Here a dozen or more camels were tethered, there children played noisily. The smells of cooking wafted on the dry air.

The patriarch Mustapha lay on his bed, mortally wounded by his battle with the giant snake, surrounded by his sons, daughters, their children and grandchildren, as befitted the head of a large family. One of his wives brought him some water in a goatskin. Coming at last to a decision, he dismissed them all, instructing them to send Ali the barber, his lifelong friend and confidant. Ali trimmed the beards of all bedouin who did not trim their own.

Ali entered the low tent and gave the traditional greeting. "Salaam aleikum. You require a haircut?"

"Aleikum salaam. No. Ali, I am becoming weak from loss of blood, and soon I must die," said Mustapha. He waved away the protests. "You know as I do that my body is broken beyond repair: there is no need to pretend. My only wish is that my wealth should be fairly divided among my three sons. I am very fond of them, but they are sometimes slow-witted, and it occurred to me that they should come into their inheritance only after a demonstration of their intellectual prowess."

Ali looked perplexed. "I do not understand your thinking, Mustapha."

"Among my possessions is an ancient arithmetical treatise, handed down, it is said, from the great Al-Khowarizmi himself. It tells of a wealthy merchant who possessed seventeen camels. He decreed that upon his death the eldest son was to have one half of them, the second son one third, and the third son one ninth."

"I remember some such conundrum. Of course, it makes no sense to offer the eldest son eight and a half camels."

"Nor the youngest one and eight ninths. Even though, as the sage wrote, it is the last straw that breaks the camel. No, the manner of the resolution is more ingenious."

"Yes, I remember. A wise man brings an extra camel of his own, raising the total to eighteen. The eldest son takes half of that number, namely nine camels; the second son takes one third, or six camels; and the youngest one ninth, or two camels. Those numbers total seventeen, whereupon the wise man departs once more with his own camel and the terms of the bequest are satisfied."

"Or at least, everybody thinks they are. The psychology of the puzzle is almost as fascinating as the mathematics."

"But Mustapha, you have more than seventeen camels."

"That is so. Indeed, Allah has blessed me with almost forty. Moreover, I promised my own father on his deathbed never to— have I told of his adventure with the forty *ifrits*? Ah, I see that I have. I was saying?— oh, yes, never to sell a camel. So it is not possible to reduce the number to seventeen. Of course, it would not be difficult to purchase a few extra camels should that prove necessary. The question that I am unable to answer is whether some other collection of numbers would permit a similarly curious course of events."

"You could always treble everything, I suppose," said Ali. "Start with fifty-one camels and the same dispostion into fractions."

Mustapha nodded again, grimacing with the pain that movement induced. "I have thought of that, Ali. But then it would be necessary for the wise man to introduce *three* extra camels. That lacks elegance."

Ali rubbed his beard. "So the question is, what other numbers of camels would behave in this curious manner."

"Yes. I had in mind assigning to each son some appropriate fraction of the total that would permit the introduction, and subsequent removal, of just one extra camel."

Ali leaned back and smiled. "Numbers, Mustapha, were always a strong point of mine. I wonder..." He gazed into space for a few seconds. "By the grace of Allah, there may be a way. But first we must understand how the original trick works."

Mustapha scratched his head. "I confess I am sorely puzzled. The entire process seems decidedly strange. The crucial camel appears and vanishes like a *djinn* from a lamp with a defective wick."

"It must be some quirk of the particular fractions chosen," said Ali. "For example, had there been twelve camels, with the sons getting one half, one third, and one sixth, then the eldest would get six camels, the second four, and the third two. No extra camel would be needed... Aha! I believe I see a ray of light. The three fractions cannot possibly add up to unity! If they did, such a trick would never work — for all camels would be divided, including the additional one, with none left over. Let me see... what is the sum of 1/2, 1/3, and 1/9? It would seem to be... ah, 17/18. Of course! Now it becomes clear. The fractions account for only 17/18 of the total. With an extra camel, the total is 18, so the fractions account for 17 camels, and the extra one can be removed again.

"Thus the crux of the trick is that the sum of the three fractions assigned to the sons must be a fraction whose denominator exceeds its numerator by one. Here the numerator is 17 and the denominator 18, which is 1 bigger. On this numerical difference does the trick hinge." He grinned broadly. "There are many such fractions... they take the form (d-1)/d for any d. I have it! How many camels do you have?"

"Thirty-nine."

"Then all we need do is choose fractions that sum to 39/40. For example 1/2, 1/4, and 9/40." He turned in triumph, but then his face fell. "You seem unimpressed, Mustapha."

"It lacks elegance, Ali. Each fraction should be *one* out of something."

"Pardon?"

"*One* third, *one* seventh, *one* nineteenth. Like that. Not *nine* fortieths."

"Ah. You require numerators of unity."

"Precisely."

"In short, you require a solution in whole numbers to the equation $1/a + 1/b + 1/c = (d-1)/d$. The number $(d-1)/d$ must be expressed as a three-term *Egyptian fraction* — a sum of three reciprocals. For so did those ancients write such numbers."

"I would bow to your superior judgement, but any such motion is beyond me."

"Nonsense! You're just as good at *al-jabr* as I am."

The patriarch chuckled. "I may even be a better algebraist, Ali. I would write your equation more elegantly:

$1/a + 1/b + 1/c + 1/d = 1$."

Ali smote his thigh in delight. "Elegant indeed, Mustapha, though no Egyptian would have wished to expressed a simple unity in such a fashion! And the ancient puzzle provides one solution:

$$1/2 + 1/3 + 1/9 + 1/18 = 1.$$

Magnificent!" He gave a conspiratorial wink. "So now all we have to do is find some other solutions to your four-term Egyptian equation. That is, find four numbers whose reciprocals sum to 1." He paused. "A reciprocal agreement, so to speak."

Mustapha's brow furrowed in thought. "I can certainly think of one other solution," he said. "Namely, 1/4 + 1/4 + 1/4 + 1/4 = 1. Though I'm not sure I see how it fits the pattern of inheritances."

"Ah, that is easy," said Ali. " 'There was once a merchant who owned three camels. He decreed that when he died, one quarter should go to his eldest son, one quarter to the second son, and one quarter to the youngest son.' "

"Ludicrous!"

"Indeed. 'So a wise man brought a fourth camel of his own, each son took one, and the extra camel was removed again.' It fits... but it's not a very convincing tale. And," added Ali, "you have more than three camels, so I regret it does not help."

The patriarch shrugged. "So what now?"

"My friend, we shall solve your equation *completely*. We shall find *all possible* solutions!" Ali reached for a sheet of paper. "It is a delicate matter, for what we have is what mathematicians call a Diophantine equation, one that must be solved using only whole numbers. Indeed, in this case, positive whole numbers. Such equations were discussed by a Greek, Diophantus of Alexandria, around the fifth century."

Mustapha turned himself awkwardly in his bed to ease his shattered bones. "Are you not being over-ambitious, Ali, to seek *all* solutions? Might there not be rather a lot?"

Ali shrugged. "Diophantine equations tend not to have very many solutions. Well, there are exceptions, but on the whole that's true. And in this case..." his voice trailed off and he began scribbling on the paper. "I believe we can prove that there exists only a finite number of solutions. Moreover, the proof allows us to *find* them all in a systematic manner. Among them may be one that suits your problem.

"Your solution with four quarters suggested the idea to me. Suppose that the numbers are arranged in order of size, so that $a \le b \le c \le d$. Then a must be at most 4. Otherwise the sum would be smaller than 1/5 + 1/5 + 1/5 +1/5 = 4/5, which it is not."

Mustapha stared at him. "This helps?"

"Oh, yes. You see, we also know that all four numbers must be at least 2. Otherwise the sum begins 1/1 + ... and becomes too large. Therefore there are only three cases to consider further: a = 2, a = 3, and a = 4. Already we have settled the possibilities for the smallest number a.

"In the first case, where a = 2, the problem now reduces to solving 1/2 + 1/b + 1/c + 1/d = 1, that is,

$$1/b + 1/c + 1/d = 1/2.$$

When a = 3 the solution reduces to a similar equation

$$1/b + 1/c + 1/d = 2/3,$$

and when a = 4 the solution reduces to

$$1/b + 1/c + 1/d = 3/4.$$

Brilliant!"

Mustapha looked puzzled. "But Ali, all you have done is replace one equation with three."

"Yes, Mustapha — but now each has only three unknowns instead of four! Moreover, I can repeat the same trick on each. For example, consider the first of the three equations, 1/b + 1/c + 1/d = 1/2. It is apparent that the second smallest number, b, must be no more than 6. Otherwise the sum would be less than or equal to 1/7 + 1/7 + 1/7 = 3/7, which is smaller than 1/2. In the same manner, for three reciprocals that sum to 2/3, b must be at most 4; and for a sum of 3/4, b must also be at most 4. Thus each of the three cases for the number a breaks up into a finite number of subcases for b."

"And then," said Mustapha in excitement, "you use the same trick yet again!"

"Precisely. As I have said, if 1/b + 1/c + 1/d = 1/2 then b must be at most 6. Since a = 2 is the smallest value, b is either 3, 4, 5, or 6. Suppose, for example, that b is 3. Then 1/2 + 1/3 + 1/c + 1/d = 1. That is, 1/c + 1/d = 1/6."

"From which," cried Mustapha, "we deduce that c is at most 12, since 1/13 +1/13 = 2/13 which is smaller than 1/6."

"Indeed, my friend. And that gives only a finite set of sub-sub-cases for c; after which d has a unique value which we can calculate precisely. For example, if a = 2, b = 3, and c = 11, then d must satisfy 1/2 + 1/3 + 1/11 + 1/d = 1, which implies that d = 66/5. But that is not a whole number, so there is no solution with a = 2, b = 3, c = 11. On the other hand, if a = 2, b = 3, c = 10, then 1/2 + 1/3 + 1/10 + 1/d = 1, which implies that d = 15. This time a solution appears. In general, if d turns out to be a whole number, then we have found a solution; if not, then that particular sub-sub-case does not lead to any solution.

"Moreover, the same argument applies to any equation of the form 1/a + 1/b + ... + 1/z = p/q, where a, b, ..., z, p, and q are positive whole numbers. There are only finitely many ways to write any given fraction as an Egyptian fraction with a fixed number of terms. The solutions can be found by successive reductions based upon the possible limits to the smallest number among those whose reciprocal is being taken."

Mustapha coughed, then spat blood. "You seem to have proved a very general theorem, Ali."

"Precisely. Now, allow me a few moments' calculation, and I shall find all possible solutions to your equation." There were no sounds save the scribbling of Ali's pen and a few distracted grunts. "I find exactly... fourteen different solutions." (See Figure 1 overleaf, and Table 1).

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Table 1 All solutions in positive whole numbers to 1/a + 1/b + 1/c + 1/d = 1, arranged so that a ≤b≤c≤d.

2	3	7	42	
2	3	8	24	
2	3	9	18	
2	3	10	15	*
2	3	12	12	
2	4	5	20	
2	4	6	12	
2	4	8	8	
2	5	5	10	
2	6	6	6	
3	3	4	12	
3	3	6	6	
3	4	4	6	*
4	4	4	4	

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"My brow is furrowed by misunderstanding," said Ali after a long period of silence. "I do not see how (2,3,10,15) can be a solution. How can 15 camels be divided into 2 or 10 parts?"

"Ahhhh... There is of course one further *arithmetic* condition, which in my haste to complete the *al-jabr*ic part of the work I have omitted. Finally, each solution of the equation must be checked to ensure that d is divisible by each of a, b, and c. Two solutions [marked by *'s in the table] fail this condition and should be removed."

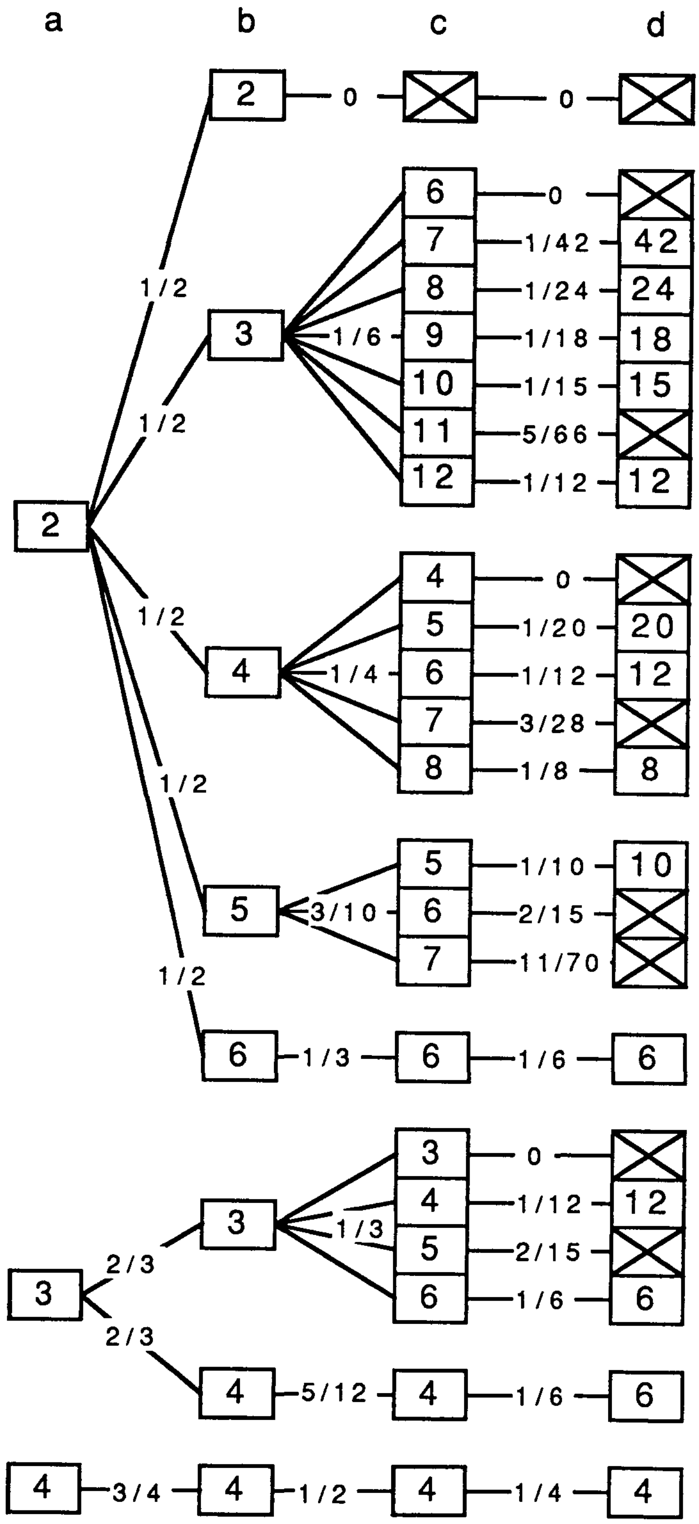

Fig.1 Ali's proof that there are exactly 14 solutions to Mustpha's equation. The columns show possible values of the numbers a, b, c, d, and lines show which combinations might occur, based upon estimates of the possible sizes of the numbers involved. Fractions written on lines show how much of the total remains at each stage, and determine the limits on the numbers that follow. Crosses show cases that lead to impossibilities.

"I am satisfied. And now the manner of your bequest stares us in the face," Ali pointed out. "The very first solution in the table is

$1/2 = 1/3 + 1/7 + 1/42 = 1$.

And the story that it tells us is this:

" 'A certain merchant possessed 41 camels. He decreed that his eldest son should inherit one half of them, his second son one third, and his third son one seventh. All were sorely confused, for what use is five and sixth sevenths of a camel? But one day a wise man brough in an extra camel, making 42 in all. Then the eldest son took 21, the second son 14, and the third son 6, total of 41. The wise man departed once more with his own camel, and all were satisfied — though even more sorely perplexed.' "

"He wasn't really a wise man, was he? He never pointed out to anyone that the fractions don't add up."

"In that omission lay his *deepest* wisdom, Mustapha."

The dying man clasped the barber's hand urgently. "Ali, you have answered my prayers. It merely remains for me to procure two more camels. Have the terms of the bequest drawn up immed— "

There was a commotion outside the tent, and a great deal of scuffling. Suddenly a small boy shot in through the flap. The patriarch fixed him with a firm but kindly stare. "Yes, Hamid? Do you normally approach the head of your family in such a precipitate fashion? Especially when he lies mortally wounded from a famous victory?"

"It is about your family that I bring tidings, Mustapha ibn Mokhta. Come with me, quickly!"

"Young man, I could as easily fly to the Moon."

"But I have important news! Your third wife, Fatima has just borne you a son! Your fourth son!"

Mustapha Looked at Ali. Ali looked at Mustapha. "*Four* sons," he said. He screwed up the sheet of paper and threw it to the ground. "It is the will of Allah," he stated. "Do not despair, Mustapha. It remains only to repeat the calculations, but this time seeking all solutions to the equation $1/a + 1/b + 1/c + 1/d + 1/e = 1$. Let me see... The smallest number a must now be no more than 5. If it is 2... Oh, beautiful! All I need do is double every number in the previous solutions and append $a = 2$ at the front. For example, since $1/2 = 1/3 + 1/7 + 1/42 = 1$, then $1/4 + 1/6 + 1/14 + 1/84 = 1/2$, so that $1/2 + 1/4 + 1/6 + 1/14 + 1/84 = 1$. If $a = 3$, however, more thought will be needed..."

Mustapha leaned forward, ignoring the pain, and tugged urgently at his friend's sleeve. "Be quick," he said. "I have many wives."

Shahryar's eyes shone with excitement. Men, *thought Scheherezade.* So easily distracted that it is scarcely flattering to one's feminine wiles...

"And what divisions of camels did Ali discover when there are four *sons, my petal?"*

"Unfortunately you will never know, my Lord," replied Scheherezade. "For it is now dawn, and my executioner awaits."

Shahryar looked thoughtful. "Well," he said, "I suppose, just this once..."

FURTHER READING

Richard K Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York 1981.

Eugene P.Northrop, *Riddles in Mathematics*, Penguin Books, Harmondsworth 1960.