Portfolio theory and the Capital Asset Pricing Model

Professor Raghavendra Rau

13 November 2023

Introduction

My name is Raghavendra Rau and I'm a professor at the University of Cambridge. This is the second in a series of Gresham lectures this year on the big ideas of finance. The lectures this year are drawn from my textbook A short introduction to corporate finance, published by Cambridge University Press.

As I said last time, there are only six major ideas in finance, five of which have won their originators Nobel prizes in economics. What are these ideas?

1. **Net Present Value (NPV):** NPV is the key principle in investment decision-making, where the objective is to maximize the present value of future payoffs. It involves three steps: computing cash flows, discounting those flows to a single present value using a discount rate, and deciding the financing method, which affects taxes.

2. **Portfolio Theory and the Capital Asset Pricing Model (CAPM):** This is the idea we are discussing in this lecture. It draws on Chapter 3 of my book. The interest rate or discount rate in NPV is determined by investors, based on available investment opportunities. Markowitz and Sharpe, Nobel laureates, proposed that individual investments are parts of portfolios. They combined portfolios with risk-free assets to determine the market portfolio. The discount rate is determined using the CAPM formula.

3. **Capital Structure Theory:** The next idea explains how the discount rate changes based on the firm's financing decision – whether to go with debt or equity. Modigliani and Miller, Nobel winners, posited that in a perfect world, financing form doesn't affect firm value. But with real-world imperfections (like taxes), it does matter. We will cover this in lecture 3.

4. **Option Pricing Theory:** Lecture 4 discusses how to price options, which are contracts that give rights to buy or sell assets. Black, Scholes, and Merton, with the latter two winning a Nobel, provided a solution based on the no-free lunch principle. They matched the cost of a portfolio replicating an option's payoff to the option's cost.

5. **Asymmetric Information:** Lecture 5 deals with information imbalances in transactions, where one party has more information than the other. Akerlof, Spence, and Stiglitz, Nobel laureates, developed key concepts in this area, illustrating how information imbalances affect markets from used cars to financial policies.

6. **Market Efficiency:** The last lecture discusses how markets reflect all available information. The debate lies in the relationship between market prices and NPV. Three Nobel winners, Kahneman, Fama, and Shiller, contributed pioneering ideas on this topic, discussing market behavior and efficiency.

In essence, corporate finance revolves around six central ideas, with five of them recognized by Nobel Prizes.

Why do we need portfolio theory?

In the first lecture in this series, we discussed the major inputs into the NPV formula. There were two obvious inputs - the cash flows and the discount rate. There is another non-obvious input – the capital structure of the firm and we will discuss this in the third lecture. How do we get the discount rate? What *is* the discount rate? These are the questions we will answer in today's lecture.

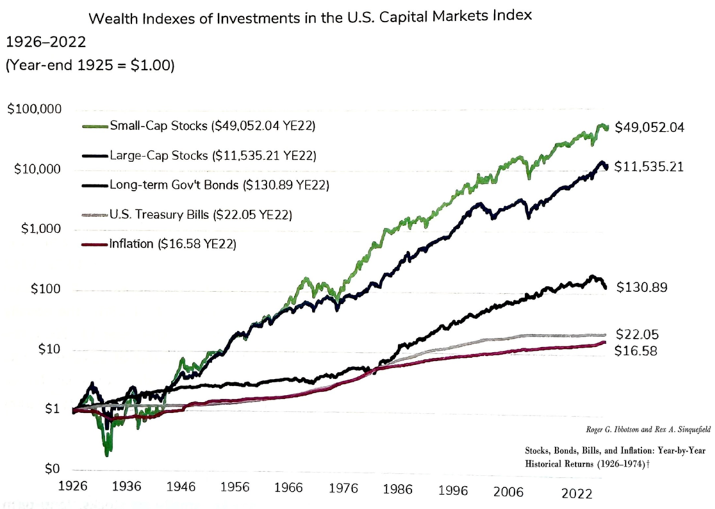
The idea is simple. Someone comes to you and asks you to invest in their company. You need to ask yourself

– if I *don't* invest in this company, what can I do with my money? We need to figure out what is the opportunity cost of investing in this company. The opportunity should have the same amount of "risk".

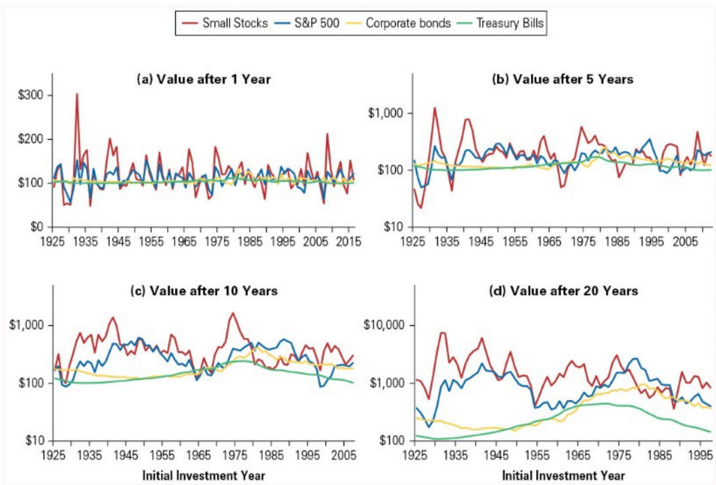
But what is risk? Does it depend on how risk-averse you are? If you hate risk, will you have a higher discount rate than someone who is risk-tolerant? Does it depend on the characteristics of the asset? What is worse is that because there is only one price for an asset, there must be only one discount rate. In other words, the alternative next-best opportunity must be the same for everyone, regardless of their risk-preferences or any other characteristics. For a long time, no one knew how to figure out a unique alternative asset for any given investment opportunity *that would be the same for everyone in the whole world*.

What does history tell us?

History tells us that if you invested $1 in 1926 in a bunch of different assets, you would have varying amounts today.

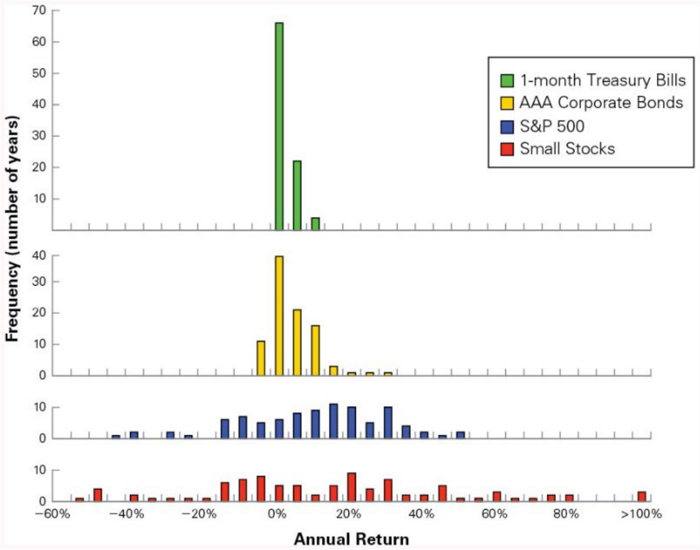


For example, investing $1 in small cap stocks would give you $49,052 today, while investing in US treasury bills would only give you $22.05. Suppose you have a time machine and can go back to 1926. Knowing this information, what would you choose to invest in? A simple rational answer might be to invest in small-cap stocks. But doing so would have lost you 92% of your money less than 2 years later as the US tumbled into the great Depression. Only in the extreme long-term would your portfolio have done so well. So does your answer depend on your horizon? Well, yes, partially.



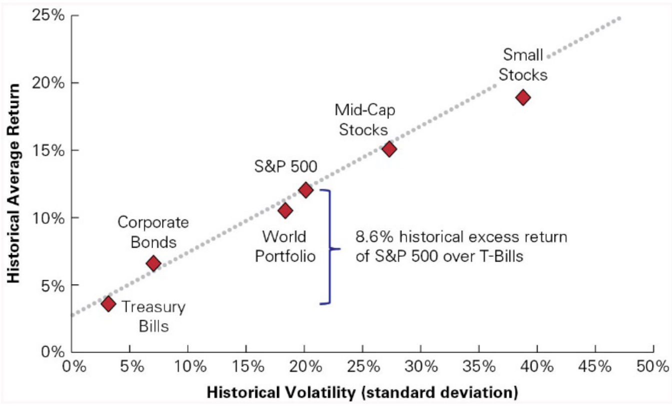
According to history, investing in stocks would only be good over very long horizons. Over shorter horizons you might lose money. Unfortunately, the future need not be like the past. We can't even explain the past. For example, in October 1987, the stock market lost 23% of its value in one day! Why? We don't know. Will it happen again? We don't know. Unfortunately, this means that from history, we cannot tell what the next best alternative investment is.

So what can we tell from the past? Well, we can see the distribution of returns.

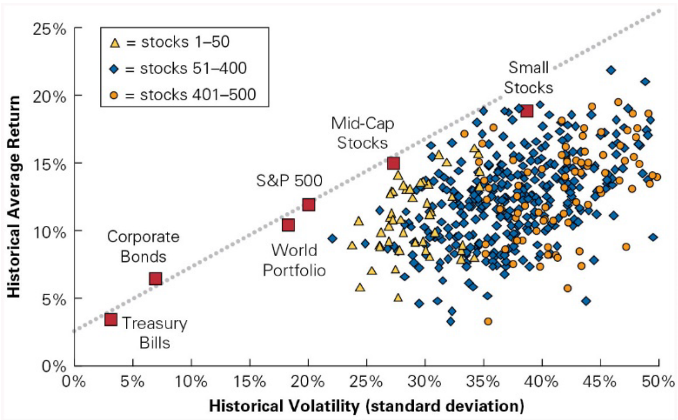


What does this figure tell us? Short-term government bonds don't make much money, but you don't lose money either (they are risk-free). With small stocks, you have the possibility of making a lot of money and the possibility of losing a lot of money. That is, the width of the distribution is much larger for small stocks than it is for short-term government bonds.

You might think that if we plot these returns against the standard deviation of the portfolios, we can get a predictable relationship between risk and return.



Unfortunately, this only works for portfolios. For individual assets, there is no relationship.



How can we measure risk?

So what does theory say? Unfortunately, we don't really know what risk is. But finance academics typically focus on variance and standard deviation. Why? That is because of something called the normal distribution. A lot of things in nature follow a normal distribution – weights, heights, etc, all follow a normal distribution. Consequently, we know a lot about this distribution. In particular, if we know the expected return (the centre point of the distribution) and the standard deviation (the variance), you know everything there is to know about the distribution. Do stocks follow a normal distribution? Not quite. The true distribution is fat-tailed. Bad (and good) stuff happens a little more frequently than predicted by the normal distribution. But most of the time, we can sort of pretend it follows a normal distribution.

Computing expected return and standard deviation for one security

So we need to compute the expected return and standard deviation for an individual security. How do we do that?

The expected return represents the average return we anticipate on an investment, given the possible outcomes and their respective probabilities.

Suppose you are analysing a security with the following potential outcomes:

Scenario	Probability	Return
Bullish	0.4	12%
Neutral	0.3	5%
Bearish	0.3	-8%

$$E(R)=(0.4\times0.12)+(0.3\times0.05)+(0.3\times-0.08)=0.039=3.9\%$$

The standard deviation measures the dispersion or volatility of the possible returns around the expected return. It gives investors an idea of the risk associated with a particular security.

Given our expected return of 3.9%, we can compute the standard deviation as:

$$\sigma=\sqrt{}(0.4\times(0.12-0.039)^2)+(0.3\times(0.05-0.039)^2)+(0.3\times(-0.08-0.039)^2) = 9.52\%$$

The expected return gives us an average return we can anticipate from the security, considering the probabilities of various scenarios. The standard deviation helps us gauge the risk or volatility associated with the security. A higher standard deviation indicates greater volatility and, hence, potentially higher risk.

Computing expected return and standard deviation for a portfolio

What about for a portfolio of two assets? Well, we can compute the expected return of the portfolio as a weighted average of the expected returns of the individual assets in the portfolio. The weights are the proportion of your wealth you have invested in each asset.

But to compute the variance of the portfolio, you need something else – the covariance between the assets. The covariance is a crucial concept when dealing with portfolios containing multiple assets. It provides insight into how two assets move relative to each other. A positive covariance indicates that the assets tend to move in the same direction, while a negative covariance suggests they move in opposite directions. For example, consider two assets, A and B, with the following potential outcomes:

Scenario	Probability	Return of Asset A	Return of Asset B
Bullish	0.5	10%	8%
Bearish	0.5	-5%	-3%

Do these assets move in the same or opposite directions? Obviously opposite directions. But how do we quantify this?

First, compute the expected returns:

$$E(R_A)=(0.5\times0.10)+(0.5\times-0.05)=0.025 \text{ or } 2.5\%$$

$$E(R_B)=(0.5\times0.08)+(0.5\times-0.03)=0.025 \text{ or } 2.5\%$$

Now, compute the covariance:

$$Cov(R_A,R_B)=(0.5\times(0.10-0.025)\times(0.08-0.025))+(0.5\times(-0.05-0.025)\times(-0.03-0.025))$$

$$Cov(R_A,R_B)=0.0056875$$

If we have the covariance, then the variance of a portfolio is given by a complicated looking formula that goes like this:

$$\sigma_P{}^2 = (x_S\sigma_S)^2 + (x_B\sigma_B)^2 + 2(x_S)(x_B)\sigma_{S,B}$$

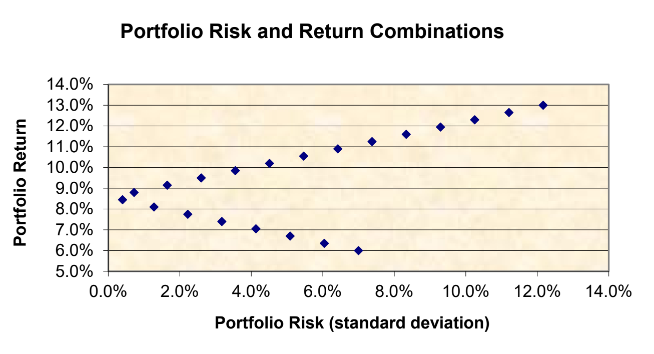
The intuition behind this formula is surprisingly simple. Consider a sack that holds an elephant and a mouse.

How much will this sack move? The formula says that it will depend on how much the elephant moves (and a small movement of the elephant will move the sack a lot because the elephant is a huge part of the portfolio), how much the mouse moves (and even a large movement of the mouse will not move the sack much because the mouse is a tiny part of the portfolio), and how much the two move relative to each other.

$$\sigma_{Sack}{}^2 = \left(size_{Elephant}\sigma_E\right)^2 + (size_{Mouse}\sigma_M)^2 + 2(s_E)(s_M)\sigma_{E,M}$$

The key though is very simple. If the covariance is negative (the two animals are moving in opposite directions (assume that this is a super-large mutant mouse)), the sack will not move at all. So, a negative covariance means that the variance of the sack will go down.

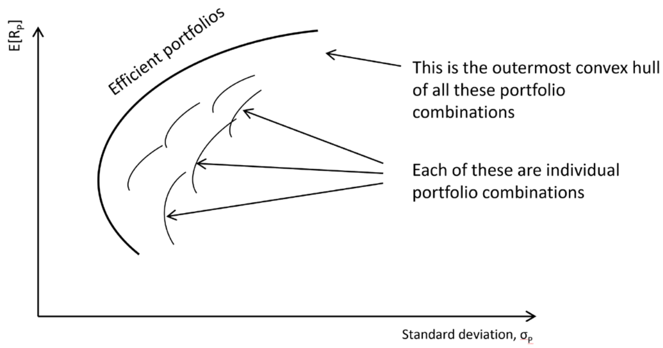
If we run these two formulae (expected returns and standard deviations) for different portfolio combinations, we will end up with a curved function that looks roughly like this.



The important thing in this graph is that no one will want to invest in the lower part of the graph. You are always better off in the upper part of the graph because you get the highest returns for the minimum amount of risk. These portfolios are called efficient portfolios.

What about a many securities world?

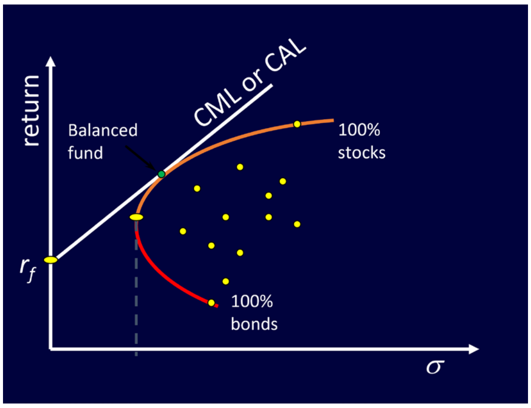
We can easily generalize this result to many securities. In a many securities world, we plot every possible combination of every possible security to get the gamut of portfolio combinations across the universe of securities. And then we take the outermost convex hull of all these combinations. That is the final (efficient) frontier. What Markowitz said is that every smart investor should invest in an efficient portfolio because that gives you the highest return for the minimum amount of risk. Therefore, when we try to find the next best alternative to the investment opportunity we are being offered, the next best opportunity must be efficient.



Adding a risk-free asset

But which one? There are an infinite number of efficient portfolios. So really Markowitz had not solved the problem completely. Sharpe noticed that there was one asset missing from Markowitz's analysis. It was the risk-free asset. Every combination of the risk-free asset and any other asset must lie on a straight line because (1) the standard deviation of the risk-free asset is always zero and (2) the covariance of the risk-free asset with anything is always zero.

So if you add a risk-free asset to the portfolio frontier, we can draw a tangent line between the risk-free asset and portfolio that will leave you better off than investing in any other efficient portfolio.



What is the equation of this line? Well, using high-school trigonometry,

$$y = intercept + slope \times x$$

We can show that the line equation is

$$E[R_P] = r_f + \frac{E[R_M] - r_f}{\sigma_M} \times \sigma_P$$

This is one version of the CAPM. However, this works only for portfolios. To get it for individual securities, we need to note one last point.

Deriving the capital asset pricing model

When computing the variance of a portfolio, we added a whole bunch of covariance terms and very few variance terms. So what is important in determining the movement of a portfolio is the covariance, not the variance.

But what is everyone holding? According to the reasoning above, everyone is holding a combination of the tangency portfolio and the risk-free asset. So, we define our ultimate measure of risk, the beta, as the *covariance* of our asset with that tangency portfolio, what we call the market portfolio, standardized by the variance of the market.

And from this, we can derive the Capital Asset Pricing Model as

$$E[R] = R_f + \beta(R_m - R_f)$$

The most important theoretical assumption we made to derive the CAPM is that investors are most concerned with mean returns and variances. If that assumption is correct, then the CAPM will tell us what the expected return is to any asset. That expected return is the discount rate we need in the NPV formula that we discussed last time.

But the story does not end here. The risk (now solely referring to beta) of the firm is also affected by leverage. To see why we need the third major idea of corporate finance – capital structure theory - but that is a subject for next time.

References and Further Reading

Markowitz, Harry, 1952, Portfolio Selection, *Journal of Finance* 7, 77-91.

Rau, Raghavendra, “A short introduction to corporate finance”, 2017, Cambridge University Press (Chapter 3).

Sharpe, W.F. (1964), Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19: 425-442. https://doi.org/10.1111/j.1540-6261.1964.tb02865.x