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**Hamilton, Boole and their Algebras**

Professor Raymond Flood

Thank you for coming to my lecture today. The topic of today’s lecture is about how the work in algebra of two nineteenth century mathematicians, based in Ireland, led to breaches of some of the fundamental laws of numbers that hold universally in arithmetic. Their work freed algebra from the constraints of arithmetic and led to the invention of new algebras.

William Rowan Hamilton revolutionized algebra with his discovery of quaternions, a non-commutative algebraic system, as well as his earlier work on complex numbers. Although George Boole contributed to probability and differential equations, his greatest achievement was to create an algebra of logic, now called Boolean algebra. These new algebras were not only important to the development of algebra itself but remain of current use — for example quaternions in computer graphics.

I will also look at some of the work of the American mathematician and engineer, Claude Shannon, who applied Boole’s symbolic logic to problems of electrical circuit switching.

I will look at Hamilton first and then at Boole. It is possible to consider them separately because although they were both based in Ireland, Hamilton in Dublin and Boole in Cork, they strangely had very little interaction or communication.

William Rowan Hamilton was undoubtedly the most important Irish mathematician who worked in Ireland. Recognition came in his own lifetime as he was elected, shortly before his death first in the list of the Foreign Associates of the newly formed American National Academy of Sciences. These Associates were in the view of the Academy the most important scientists working outside the United States of America. Although Hamilton had performed world class research in geometrical optics, as well as in dynamical systems, it was his work on quaternions which won him recognition from the National Academy.

Hamilton was born in Dublin in 1805 and was a child prodigy with the legend that by the age of thirteen he could speak as many languages. One of his biographers, Sean O’Donnell, has analysed this claim and comes up with a more modest assessment that he knew Latin, Greek and Hebrew well enough to read them easily with lesser facility to read French, German and Italian and that his knowledge of Persian, Syriac, Sanskrit and Chaldee seems to have been more modest! Interestingly O’Donnell also says that Hamilton seems to have claimed no knowledge of Gaelic, the language spoken by half the population of Ireland at the time.

Hamilton was educated by his uncle James Hamilton with an emphasis on classics – although he also showed remarkable calculating ability. But by the time he was preparing for entrance to Trinity College Dublin his preference for mathematics and science were becoming clearer as he indicates in a letter to his sister Eliza

*One thing only have I to regret in the direction of my studies, that they should be diverted – or rather, rudely forced – by the College Course from their natural bent and favourite channel. That bent, you know is Science – Science in its most exalted heights, in its most secret recesses. It has so captivated me - so seized on …. My affections – that my attention to Classical studies is an effort and an irksome one.*

He took first place in the entrance examinations for Trinity the following year, quickly scaling the academic ladder and becoming Professor of Astronomy and Astronomer Royal of Ireland at the age of 22 before he even graduated.

On this slide at the top we have a panorama taken from inside Trinity College Dublin. Dunsink Observatory below was Hamilton’s home as the Andrews' Professor of Astronomy and Royal Astronomer of Ireland. The observatory is situated on the highest point to the north-west of Dublin about 8km from the city centre. Hamilton discovered quaternions while walking to Dublin from Dunsink on the 16th October 1843 and this is celebrated annually with a commemorative walk.

However he first worked mainly on geometrical optics and dynamics.

In 1832 he obtained international recognition when he predicted that a ray of light incident on a biaxial crystal in a certain direction would be refracted into a cone of rays and emerge as a hollow cylinder. A biaxial crystal is one with certain optical symmetries.

In this slide we see the ray of light at the top incident on the crystal at a certain angle when it is refracted into a hollow cone inside the crystal and emerges as a hollow cylinder.

This theoretical prediction was experimentally verified shortly afterwards by his Trinity College colleague Humphrey Lloyd, professor of natural philosophy. It caused a sensation, being one of those infrequent occasions when a theoretical investigation predicted previously unknown physical behaviour.

Hamilton’s work was very theoretical and abstract and his formulation of mechanics, based on the principle of least action, was the one approach to classical mechanics that carried over to quantum mechanics. For many systems the function known as the *Hamiltonian* is the total energy of the system and is used in both classical and quantum mechanics to discuss the evolution of a system over time.

Hamilton was knighted during the 1835 meeting of the British Association for the Advancement of Science held in Dublin. Subsequently he was President of the Royal Irish Academy and as was mentioned was elected the first foreign member of the National Academy of Sciences shortly before his death in 1865.

Hamilton’s other great discovery was that of quaternions which he discovered in 1843 after fifteen years work, and on which he spent most of his efforts during the remaining twenty two years of his life.

I will start my discussion by looking first of all at complex or imaginary numbers.

What could the square root of minus 1 mean since every real number, positive or negative, gives a positive result when squared? So the square root of minus 1 cannot be a real number.

Euler, in the eighteenth century, who worked with this type of number a great deal, also said:

*Of such numbers we may truly assert that they*

*are neither nothing, nor greater than nothing,*

*nor less than nothing, which necessarily*

*constitutes them imaginary or impossible.*

Even in the early 19th century there was still a great deal of unhappiness about complex numbers, or so-called ‘imaginary’ numbers that do not seem to exist.

For example, Augustus De Morgan, Professor of Mathematics at University College, London, declared that:

*We have shown the symbol to be void of meaning, or rather self-contradictory and absurd.*

But however strange complex numbers seemed they were of great use in solving quadratic and other equations and in other areas of mathematics.

Suppose that we try to calculate with the symbol.

We find that addition is easy:

(2 + 3) + (4 + 5) = 6 + 8,

And so is multiplication (replacing x whenever it appears by

(2 + 3) x (4 + 5)

= (2 x 4) + (3 x 4) + (2 x 5) + (15 x x )

= (8 15) + (12 +10)= 7 + 22.

We can carry out all the standard operations of arithmetic on these new objects.

We call the object *a* + *b*a *complex number*:

the number *a* is its *real part*, and the number *b* is its *imaginary part*.

Nowadays, we usually use the letter *i* to mean , so that *i2* = 1.

In 1799 complex numbers were given a geometrical form by the Danish navigator Caspar Wessel. In this representation, called the *complex plane*, two axes are drawn at right angles (the *real axis* and the *imaginary axis*) and the complex number *a* + *b*is represented by the point at distance *a* in the direction of the real axis and height *b* in the direction of the imaginary axis.

So, what did Hamilton do? In hindsight it seems very obvious and is now generally accepted!

The geometrical representation of complex numbers by the so-called Argand diagram raised the following question in Hamilton’s mind.

*Is there any other algebraic representation of complex numbers that reveals all valid operations on them?*

He proposed that the complex numbershould be *defined* as a pair (*a*, *b*) of real numbers.

We combine such pairs (*a*, *b*) and (*c*, *d*) by using the following rules:

*Addition*:

(*a*, *b*) + (*c*, *d*) = (*a* + *c*, *b* + *d*);

*Multiplication*:

(*a*, *b*) x (*c*, *d*) = (*ac* - *bd*, *ad* + *bc*);

The pair (*a*, 0) then corresponds to the real number *a*. The pair (0, 1) corresponds to the imaginary number *i =*

, and we have the equation

(0, 1) x (0, 1) = (-1, 0),

which corresponds to the equation *i*  x *i* = - 1.

Hamilton avoided talking about imaginary quantities by dealing with pairs or couples of real numbers and giving formal rules for adding and multiplying these couples. These rules then gave an exact parallel to the rules for adding and multiplying complex numbers but without introducing the mysterious square root of minus 1.

The logical development was to consider number triples or *triadic fancies* as Hamilton called them.

He now looked at number triples such as (*a, b, c*) and wanted to find rules for their addition and multiplication.

By analogy with complex numbers he wrote them as

*a + bi + cj* where *i2* = *j2 =* -1.

Adding triples was easy, for example:

*(1 + 2i + 3j*) *+ (4 + 5i + 6j) = (1 + 4) + (2i + 5i) + (3j + 6j)*

*= 5 + 7i + 9j*

But what about multiplying triples? What is, or rather how could we define:

*(1 + 2i + 3j*) *x (4 + 5i + 6j)?*

If we multiply them out in the analogous way to complex numbers and use *i2* = *j2 =* -1 we obtain

(4 – 10 – 18) + (5 + 8)*i +* (6 + 12)*j* + (12 + 15)*ij*

What is this term *ij*?

Hamilton thought about his problem for many years exploring different possibilities for the term *ij.*

In fact it became something of a family joke as we can read in this letter from Hamilton to one of his sons, Archibald, written many years later, shortly before his death.

*Every morning on my coming down to breakfast, your brother Wlliam Edwin and yourself used to ask me. ‘Well, Papa, can you* ***multiply*** *triplets?’ Whereto I was always obliged to reply, with a sad shake of the head ‘No, I can only* ***add*** *and subtract them*

Hamilton tried various possibilities for a value for *ij*.

Try *ij* = 1

Try *ij* = -1

Try *ij = 0*

There were others but they all failed because Hamilton wanted to construct a way to multiply triples which had a property satisfied by the real numbers and the complex numbers. This property is called the division *law*.

The division law says that the way of multiplying must allow any non-zero triple to divide into any other triple. More precisely if *D, E* are triples with *D* non zero then there must be a triple *X* so that

*D* multiplied by *X* equals *E*

We can think of this as meaning that *D* divides *E*.

Examples: 3 x = 17 and (1 + i)x(1 – i)/2 = 1

This division law is satisfied if he could make his multiplication satisfy another law – the law of the modulus.

The modulus of *a + bi + cj* is just

Hamilton wanted his method of multiplication to have the following property that the modulus of (*a + bi + cj)(x + yi + zj)* to equal the modulus of *(a + bi + cj)* times the modulus of *(x + yi + zj).*

This is often stated as the modulus of the product is the product of the moduli of the individual terms. If this were the case then the division law would follow.

And then suddenly one day he had the flash of inspiration that resolved the problem.

It was on the 16th October 1843 and Hamilton was walking along the Royal Canal in Dublin to a meeting of the Royal Irish Academy when as he later wrote:

an undercurrent of thought was going on in my mind which gave me at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark splashed forth, the herald as I foresaw immediately of many long years to come of definitely directed thought and work by myself if spared, and at all events on the part of others if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out the spot a pocket book, which still exists, and made an entry there and then. Nor could I resist the impulse— unphilosophical as it might have been—to cut with a knife on a stone on Brougham Bridge, as we passed it, the fundamental formula with the symbols *i, j*, and *k* namely *i2* *= j*2 *= k2 = ijk = -1*

which contains the solution of the problem.

Hamilton was a compulsive scribbler. According to his son, if there was no paper available, he would write on his fingernails or if at breakfast scribble on his hard-boiled egg.

Well, what did he see in his flash of inspiration? Instead of triples he added another term *k* which also satisfied *k2 = -1* giving him quadruples or as he called them quaternions.

In fact Hamilton’s quest for a way to multiply triples that satisfied the division law was doomed to failure. This was proved by the German mathematician Frobenius thirteen years after Hamilton’s death when Frobenius showed that the only possible real associative division algebras are the reals themselves, the complex numbers and the quaternions.

Instead of triples *a + bi + cj* he was now working with quadruples *a + bi + cj + dk* and which he could multiply using the relations *i2* *= j*2 *= k2 =ijk = -1.*

These relations imply that *ij* is not equal to *ji* but is equal to *–ji.*

We have *ijk = –1*

Multiply both sides by *k*

*ijkk = –k*

But *k2 = –1* so

*–ij = –k* or *ij = k*

Start again with *ijk = –1*

But this time multiply by *ji*

*jiijk = –ji*

But *i2 = –1* so

*–jjk =–ji*

But *j2 = –1*so

*k =–ji*

So *ij = k = –ji*

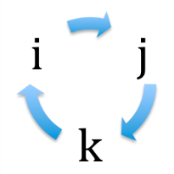
Hamilton’s great insight was to realise that he could sacrifice commutativity (which means that the order of multiplying is unimportant) and still have a consistent and meaningful algebra.

To multiply

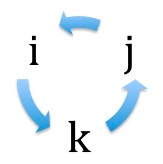
*(a + bi + cj + dk)(w + xi + yj + zk)*

We use the rules for multiplying quaternions that can be summarised as in the following diagrams:

Multiplying clockwise

Clockwise: *ij = k, jk = i, ki = j*

Multiplying anticlockwise

Anticlockwise: *ji = -k, ik = -j, kj = -i*

Quaternions satisfy all the usual arithmetical laws, apart from the commutative law. It was the first non-commutative algebra and was important in the development of abstract algebras. Hamilton was so pleased with his discovery that he wrote a poem about it.

And how the One of Time,

of Space the Three,

Might in the Chain of Symbols girdled be

Although they did not become the powerful and widespread tool that Hamilton hoped they would be, quaternions were however important in the development of vector analysis. Hamilton used the notation of ‘scalar part’ for the term – it is time in the poem - that does not involve i, j, or k, and ‘vector part’ for the terms that do – it is space in the poem.

If q1 and q2 are two quaternions with no scalar terms,

Then the “scalar” part of q1q2 is the negative of the dot product of the vectors q1 and q2,

while the “vector” part of q1q2 is the vector product of the vectors q1 and q2.

Quaternions could be used to achieve the transformation of any directed line in three dimensions to any other directed line which is why they are of use in computer graphics. They overcome various issues which affect other methods of rotating points in three dimensional space.

Let me turn now to another mathematician, the English born George Boole. The early 1840s were also an important period in his life. In 1844 the 29 year old Boole was awarded the first Gold Medal for mathematics from the Royal Society. This was an impressive achievement particularly because Boole was essentially a self-taught mathematician. Boole was born in Lincoln into a poor family; his father was a shoemaker, leaving school at age 16 and not able to afford to go to university. He taught himself languages, philosophy and mathematics. When his father’s business failed he supported his family by teaching, eventually opening his own school in Lincoln. He started to publish his mathematical researches and it was his work on applying algebraic methods to the solution of differential equations that brought him the award of the Royal Society Gold Medal.

In 1849 Boole was appointed as the first Professor of Mathematics in Ireland’s newly founded Queen’s College at Cork, now called University College Cork. He worked there until his death at the early age of 49. He married Mary Everest, a niece of Sir George Everest after whom the world’s highest mountain is named. They had five daughters, the middle one Alicia becoming a mathematician.

Boole was particularly interested in representing the workings of the human mind in symbolic form and his major work on the subject was his 1854 book *An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities*

At its start he writes:

*The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; And as the title shows he also applied his investigations to probabilities.*

He was very proud of this work writing:

*I am now about to set seriously to work upon preparing for the press an account of my theory of Logic and Probabilities which in its present state I look upon as the most valuable if not the only valuable contribution that I have made or am likely to make to Science and the thing by which I would desire if at all to be remembered hereafter*

In the *Laws of Thought* he approached logic in a new way reducing it to a simple algebra, incorporating logic into mathematics*.* He did this by applying algebraic operations on entities that had not previously been thought of as part of mathematics. These entities were classes of objects with particular properties.

For example he wrote:

Let the symbol *x* denote the class of all white things

And the symbol *y* denote the class of all sheep

Then he used the compound symbol *xy* to denote the class of all white sheep. We now think of it in terms of set theory where we interpret *xy* as the intersection of the set of white things with the set of sheep.

He continues:

And if *z* denotes the class of all horned things

then

*zxy* = all horned white sheep

Let us now consider the laws that this composition satisfies.

Again if *x* = all white things

And *y* = all sheep

then *xy = yx*

Since the class of all white things that are sheep is the same as the class of all sheep that are white. The composition law is commutative, as we have in arithmetic. But now for something different!

If every member of class *x* (say being a man) is also a member of class *y* (say being human) then

*xy = x*

and in the special case when *x* and *y* are the same

*xx = x*

or using the analogy of multiplication of numbers we can write

*x2 = x*.

This is true for **every** class and is different from what holds in arithmetic where *x2 = x* only holds for the numbers 0 and 1.

Of course it is not surprising to get results different from arithmetic as the objects we are dealing with are very different from numbers.

Boole also defines another way of combining two classes, using + for the symbol of combination.

If *x* and *y* are classes, *x + y* denotes the class of all objects which belong either to class *x* or to class *y*.

Boole in fact only defined + if x and y are mutually exclusive classes, so there is no object which belongs to class *x* and to class *y*, but latter mathematicians widened the definition to any two classes.

An example is that if x is the class of all women and y is the class of all men then x + y is the class of all humans.

Then once again we have commutativity as x + y = y + x

We have other results similar to what we see in arithmetic, for example

z(x + y) = zx + zy

If z is the class of all Europeans then the left hand side is the class of all European humans while the right hand side is the class of European women or European men.

Boole is developing an algebraic system of what we now call set theory.

Boole also used the symbol *1* to denote the universal class, the symbol *0* for the empty class and he wrote *1 – x* for the class of all objects not in *x*.

Then say *x =* all white things

So *(1- x)* = all things not white

And then since objects cannot be white and not white at the same time

*x(1 – x) = 0*

or *x – x2 = 0*

or *x2 = x* as before.

However the major impact of the *Laws of Thought* was Boole’s purely symbolic manipulation of classes.

Let me illustrate it by demonstrating the classical implication

If all As are B, and all Bs are C: therefore all As are C

Let *a* equal the class of all As

Let *b* equal the class of all Bs

Let *c* equal the class of all Cs

In Booles notation the hypotheses are

a = ab and b = bc

Then by substitution a = ab = a(bc) = (ab)c = ac so

a = ac so all As are C

I will finish by showing how Claude Shannon analysed electrical circuits using ideas similar to those of Boole.

Claude Shannon was an American mathematician, electrical engineer and cryptographer. He is possibly best known as the founder of information theory which he created in the late 1940s. I want to tell you about some of his earlier work done while he was a student at MIT. This work applied symbolic logic, which was essentially the creation of Boole, to problems in electrical circuit switching and so laid the foundations of the circuitry principles underlying the logic of computer design.

Shannon’s Master’s thesis submitted in 1936 was titled *A Symbolic Analysis of Relay and Switching circuits.* In it he looked at two general problems. The first was to analyse a given electrical circuit and in particular try to simplify it. The other problem was to try to construct an electrical circuit with given characteristics. His approach was to represent a circuit by an equation representing the structure of the circuit. He then simplifies the equation for the circuit using rules exactly analogous to those used in symbolic logic.

Let me show you how he represented an electrical circuit by an equation.

Shannon says that the *hindrance* between two points *a* and *b* on a circuit is 0 if current can flow and 1 otherwise.

If the switch between *a* and *b* is closed the hindrance is 0 and if open the hindrance is 1

The hindrance of two switches X and Y in *series* is written X + Y.

The hindrance of two switches X and Y in *parallel* is written XY.

He then sets out some basic postulates such as, for example these two:

a closed circuit in series with an open circuit is an open circuit

a closed circuit in parallel with an open circuit is a closed circuit

Then Shannon’s plus and multiplication governing hindrance satisfies certain laws which turn out to the same as Boole’s.

These laws are straightforward to prove using what he calls “perfect induction” where he considers all possible values the variables can have and applies his postulates. This is quite straightforward as each variable can only take on the values 0 or 1.

Now to see the power of this setup!

This circuit is made up of a collection of switches in series and in parallel and we can use Shannon’s notation for series and parallel to write down its equation.

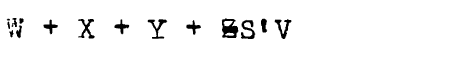
Starting from the left we have W in series so + with the next block which is W' in parallel with X and Y in series so is W' (X + Y). All of this is in series with another block of switches…

The equation for the whole circuit is:



Note the dash notation W' means a variable which has value 1 when W is 0 and 0 when W is 1.

We then use the laws to multiply out and simplify to get



Finally we use this equation to build the equivalent but much simplified circuit shown on the slide where switches W, X and Y are in series with the block of Z, S' and V in parallel.

Boole’s 200th birthday was earlier this month on the 2nd November and you may have seen that Google marked it with the Google Doodle.

This is a little animation using Boolean logic and was an appropriate tribute to Boole on his Birthday, recognising his contribution to digital computing.

Another memorial to him is in a stained glass window in Cork.

Here we see Boole at the bottom sharing the window with Aristotle and Euclid – distinguished company indeed. And on the right is the plaque on the bridge where Hamilton scratched his formula for quaternions. These memorials mark the work of two mathematicians who can fairly be said to have liberated algebra from the constraints of arithmetic.

You can find out more about Hamilton at David Wilkins fine website at

http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/

Des MacHale’s recently published updated biography of Boole tells much more about this fascinating man.

My next lecture is on Charles Babbage and Ada Lovelace on Tuesday 19 January next year.

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