The Mathematics of Game Theory

Professor Sarah Hart

Gresham Professor of Geometry

Price war

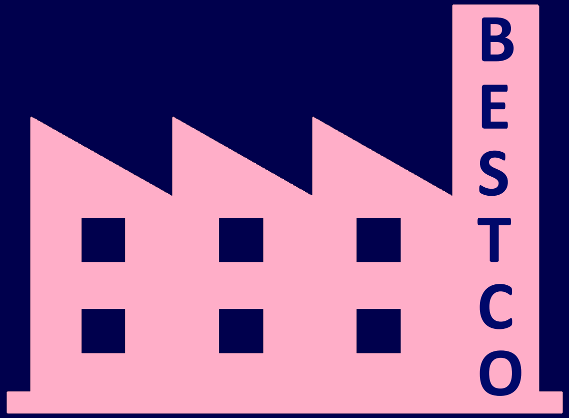
World's best coffee

The Auction; or Modern Conoisseurs.

The Prisoner's Dilemma

	Prisoner B talks	Prisoner B stays silent
Prisoner A talks	A gets 3 years B gets 3 years	A gets 0 years B gets 5 years
Prisoner A stays silent	A gets 5 years B gets 0 years	A gets 1 year B gets 1 year

Price War



- Acme vs BestCo: market for 100 widgets
- At same price each sell 50; otherwise 80-20 split
- High → £5 profit/widget; Low → £4 profit/widget

Acme vs BestCo	**B cuts price**	**B maintains price**
A cuts price	A’s profit: £200 B’s profit: £200	A’s profit: £320 B’s profit: £100
A maintains price	A’s profit: £100 B’s profit: £320	A’s profit: £250 B’s profit: £250

David Blackwell (1919-2010)

“[t]he situation with the Soviet Union has had elements like this in it. To cooperate is to disarm and to double-cross is to re-arm with bigger and bigger weapons. That takes a lot of resources and we would both be better off disarming. But each is afraid that if he throws away his weapons, the other one will not and he will be at a great disadvantage.”

The Petticoat Duellists.

Publish'd by W. & J. Stratfords, No. 112 Holborn Hill, Augt. 1. 1792.

Rules of Engagement

- One bullet each
- Stand 20 paces apart, walk 0-10 paces before firing
- Opponents equally skilled
- Probability of hitting opponent after p paces is $\frac{p}{10}$
- If your opponent fires first and misses, you win
- If both hit, or both miss simultaneously, it’s a draw.

Comparing Strategies: Colin vs Rose

- Payoff for Rose: $+1$ for win, 0 for draw, -1 for lose
- Zero-sum game
- Expected payoff: Rose to fire at 4 paces, Colin at 6

$$(+1)\times\frac{4}{10}+(-1)\times\frac{6}{10}=-0.2$$

- Rose 4 paces, Colin 2 paces:

$$(+1)\times\frac{8}{10}+(-1)\times\frac{2}{10}=0.6$$

- Rose 4 paces, Colin 4 paces:

$$(+1)\times\frac{4}{10}\times\frac{6}{10}+(-1)\times\frac{6}{10}\times\frac{4}{10}=0$$

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	Min
R1	0	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8	−0.8
R2	0.8	0	−0.6	−0.6	−0.6	−0.6	−0.6	−0.6	−0.6	−0.6	−0.6
R3	0.8	0.6	0	−0.4	−0.4	−0.4	−0.4	−0.4	−0.4	−0.4	−0.4
R4	0.8	0.6	0.4	0	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2	−0.2
R5	0.8	0.6	0.4	0.2	0	0	0	0	0	0	0
R6	0.8	0.6	0.4	0.2	0	0	0.2	0.2	0.2	0.2	0
R7	0.8	0.6	0.4	0.2	0	−0.2	0	0.4	0.4	0.4	−0.2
R8	0.8	0.6	0.4	0.2	0	−0.2	−0.4	0	0.6	0.6	−0.4
R9	0.8	0.6	0.4	0.2	0	−0.2	−0.4	−0.6	0	0.8	−0.6
R10	0.8	0.6	0.4	0.2	0	−0.2	−0.4	−0.6	−0.8	0	−0.8

Finding an equilibrium

- Two player zero-sum games, eg taking penalties

Odds of goal	Keeper →	Keeper C	Keeper ←
Striker →	50%	80%	90%
Striker C	80%	30%	80%
Striker ←	90%	80%	50%

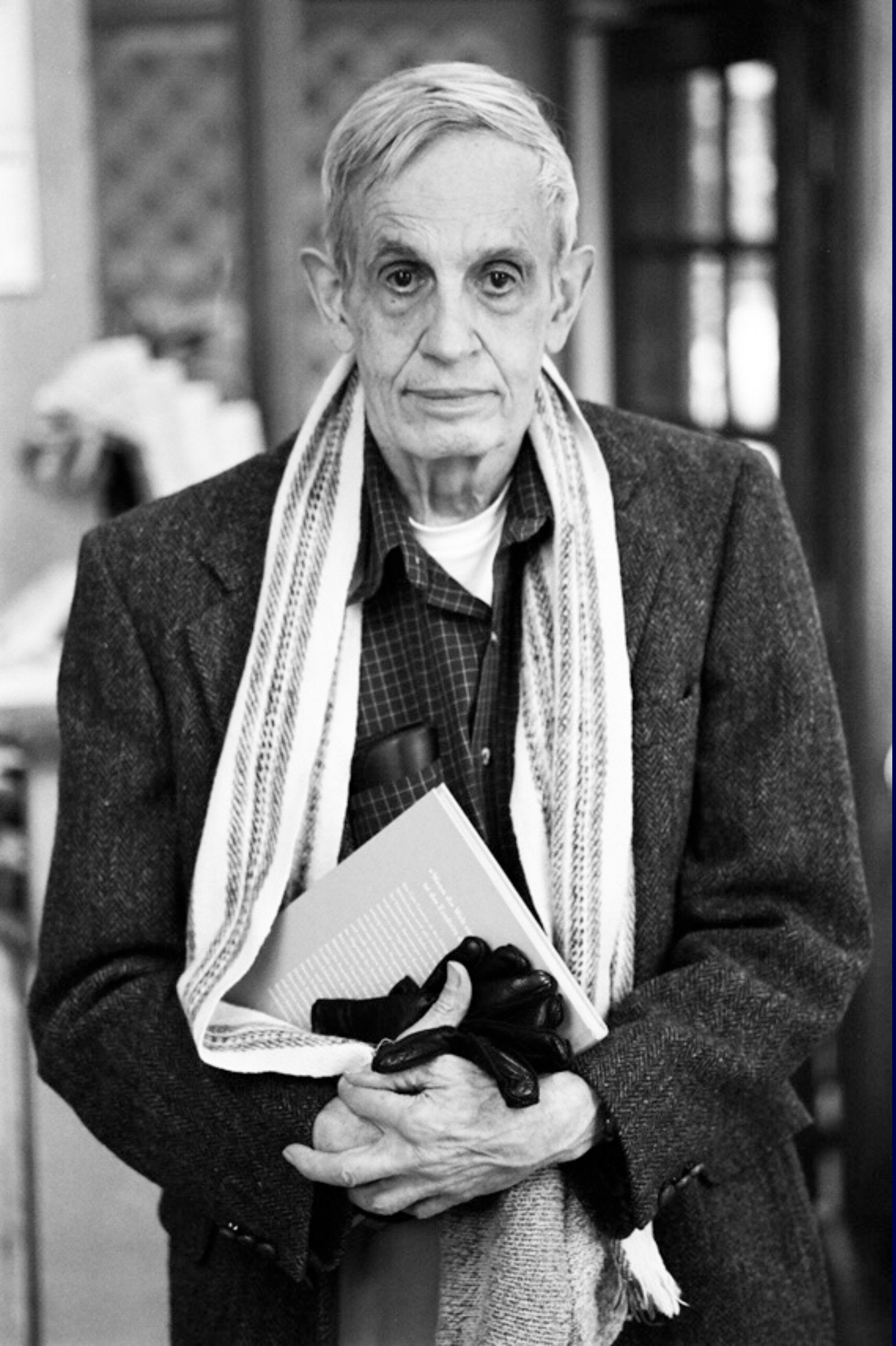
- "Pure strategy" won't work; need "mixed strategy"
- Eg players go in each direction 1/3 of the time. Expected score:

$$\tfrac{1}{9}(0.5 + 0.8 + 0.9 + 0.8 + 0.3 + 0.8 + 0.9 + 0.8 + 0.5) = 0.7$$

John von Neumann’s Minimax Theorem (1928)

- In 2-player zero-sum games there is always an equilibrium strategy that’s minimax for one player (and maximin for the other)
- Football equilibrium: both players do 42% right, 42% left, 16% centre.

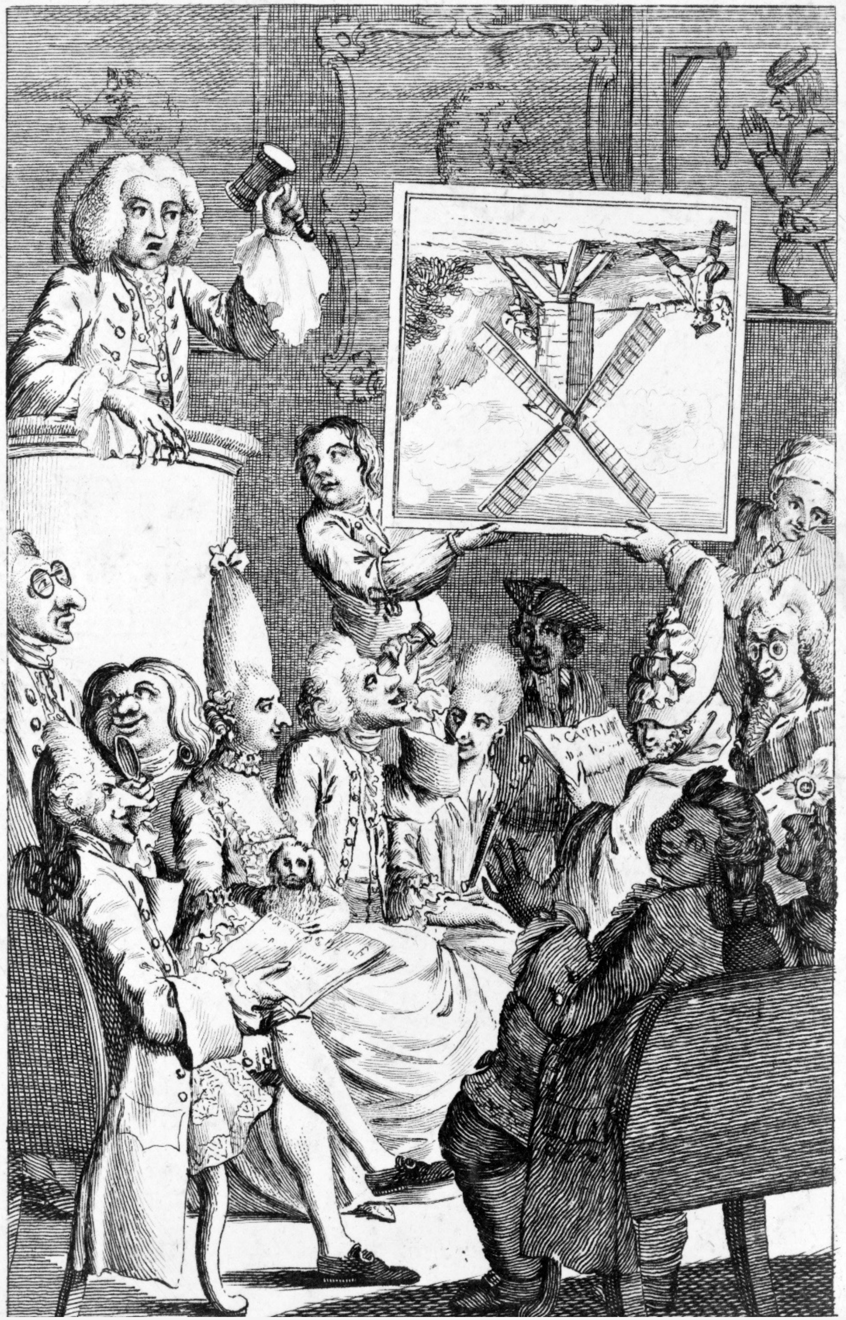
(for details see paper by Ferenc Forgó, linked in transcript)



John Nash (1928-2015)

- Multiplayer, non-zero-sum games always have a *Nash equilibrium*: no player can improve outcome by unilaterally changing strategy
- Eg “best coffee in town”: 400 price-savvy locals, 200 tourists.

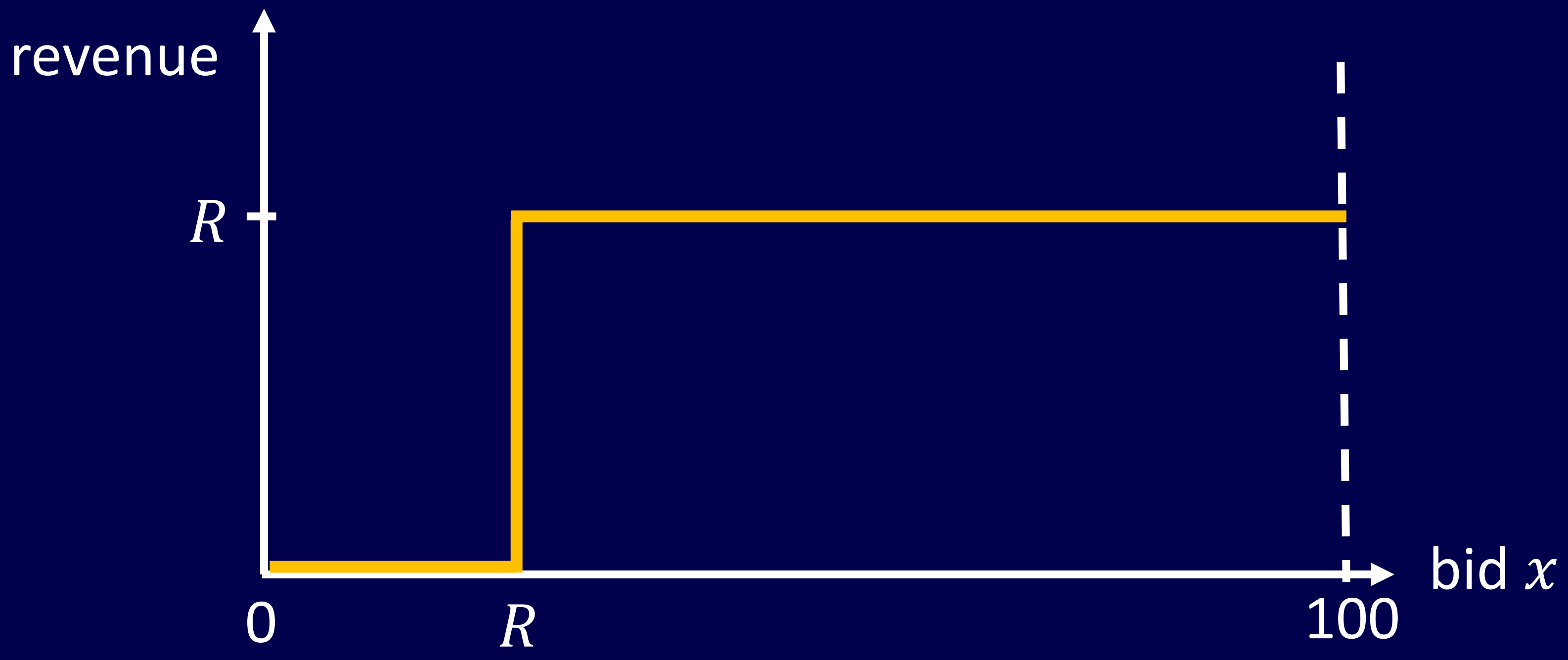
	B £3	**B £4**
A £3	A: £900 B: £900	A: £1500 B: £400
A £4	A: £400 B: £1500	A: £1200 B: £1200

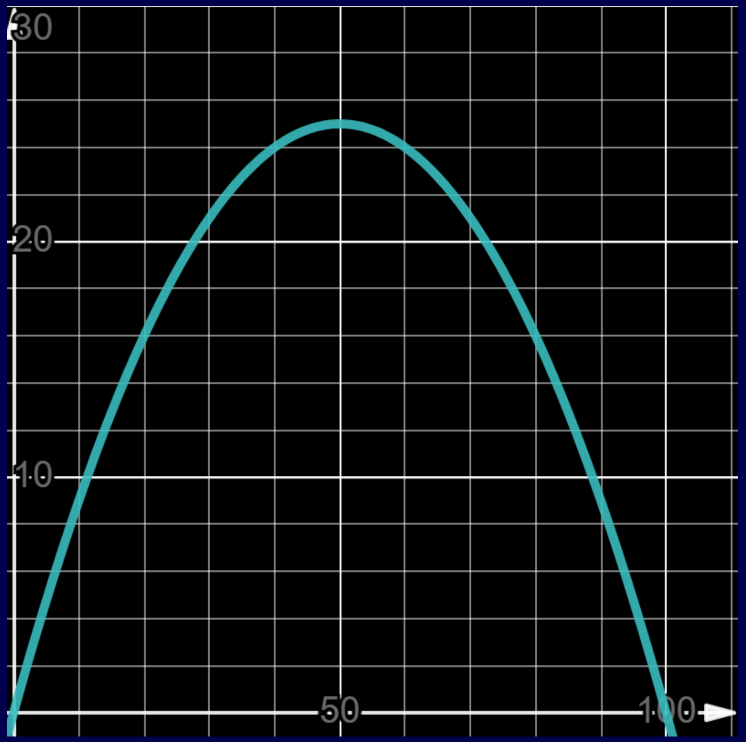


English

Setting a Reserve Price

- English auction. Reserve price R.
- Problem reduces to single bidder, max bid $x \leq$ £100.
- Expected revenue $\frac{1}{100}R(100 - R)$
- Maximum revenue at $R = 50$





What’s the best bid?

First price sealed bid auction

- Risk of winner’s curse (due to private vs common value)
- Broken lamp – can fix and sell for £100 – what bid?

Second price sealed bid auction (Vickrey auction)

- Better analogue for open English auction
- Should bid your valuation

Revenue Equivalence Theorem

- *Theoretical* expected revenue same in all four auction types
- half the highest valuation

The great 3G auction

- UK 3G licences auctioned in 2000
- 5 licences
- Round 1: make bid on one licence
- Round X: if (and only if) not top bidder on any licence, improve top bid on one licence or drop out
- Continue until 5 bidders remain
- Raised £22.5 BILLION.

Which house to rent or buy?

- Assume n houses randomly ordered.
- Strategy: view s houses out of potential n houses, then choose the first one after that which is the best so far.

$n = 3$ houses

- $s = 0$ Choose House 1 33.3% success rate
- $s = 1$ 1/6 chance of each ordering

  House 1 > House 2 > House 3 - FAIL

  House 1 > House 3 > House 2 - FAIL

  House 2 > House 1 > House 3 - SUCCEED

  House 2 > House 3 > House 1 - SUCCEED

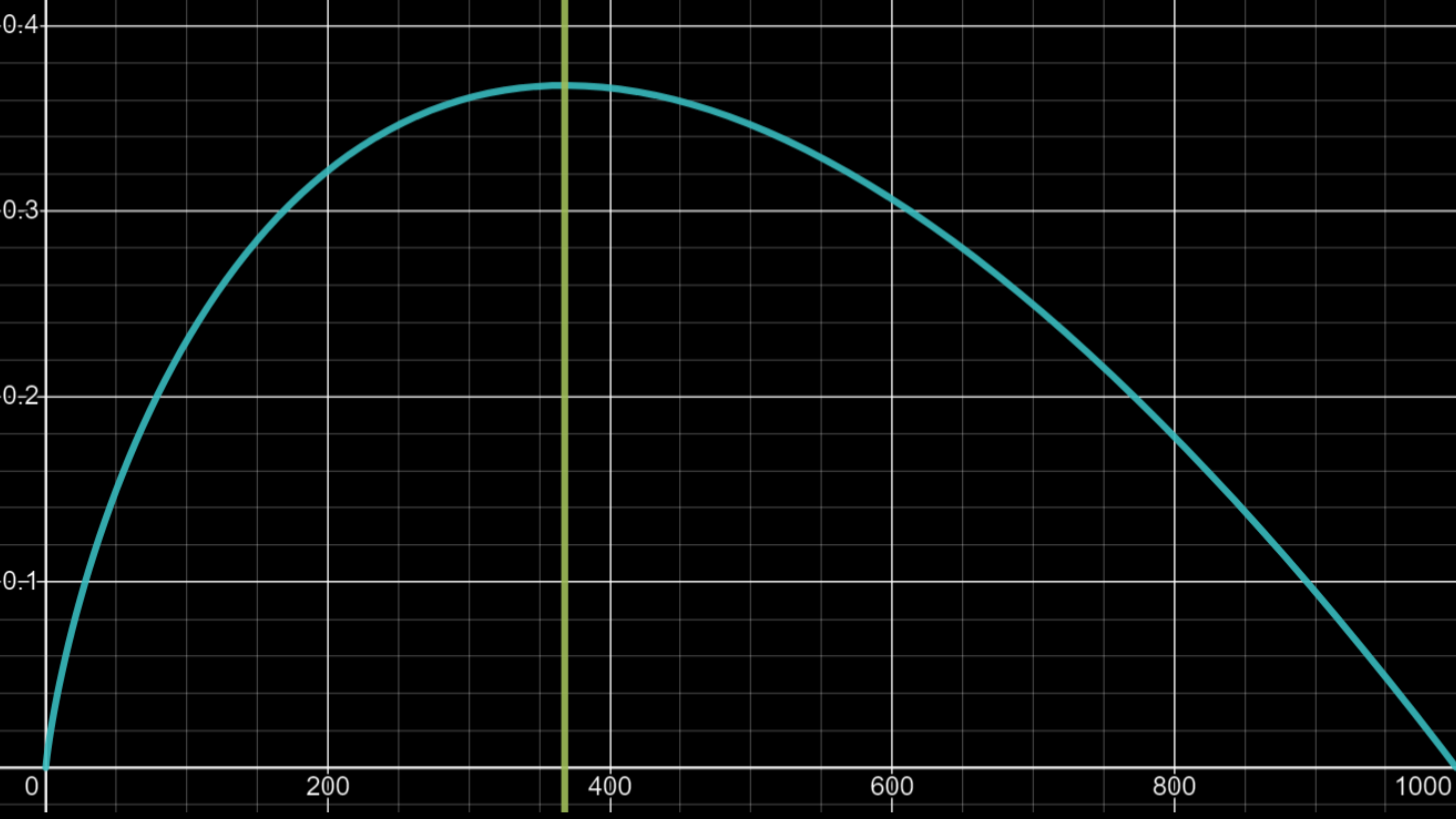
  House 3 > House 1 > House 2 - SUCCEED

  House 3 > House 2 > House 1 – FAIL 50% success rate
- $s = 2$ Reject Houses 1 & 2 33.3% success rate

Which house?

Strategy: view s houses out of potential n houses, then choose the first one after that which is the best so far. Best sample size?

Number of Houses	Best sample size	Best sample %
3	1	33.3%
5	2	40%
8	3	37.5%



0.4
0.3
0.2
0.1
0
200
400
600
800
1000

Which house?

Strategy: view s houses out of potential n houses, then choose the first one after that which is the best so far. Best sample size?

Number of Houses	Best sample size	Best sample %
3	1	33.3%
5	2	40%
8	3	37.5%
Large n		36.79%

Best sample size 36.79%, or ${}^{1}/_{e}$ of the total!

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Lottery-winning mathematics

31st January 2023, 1pm

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