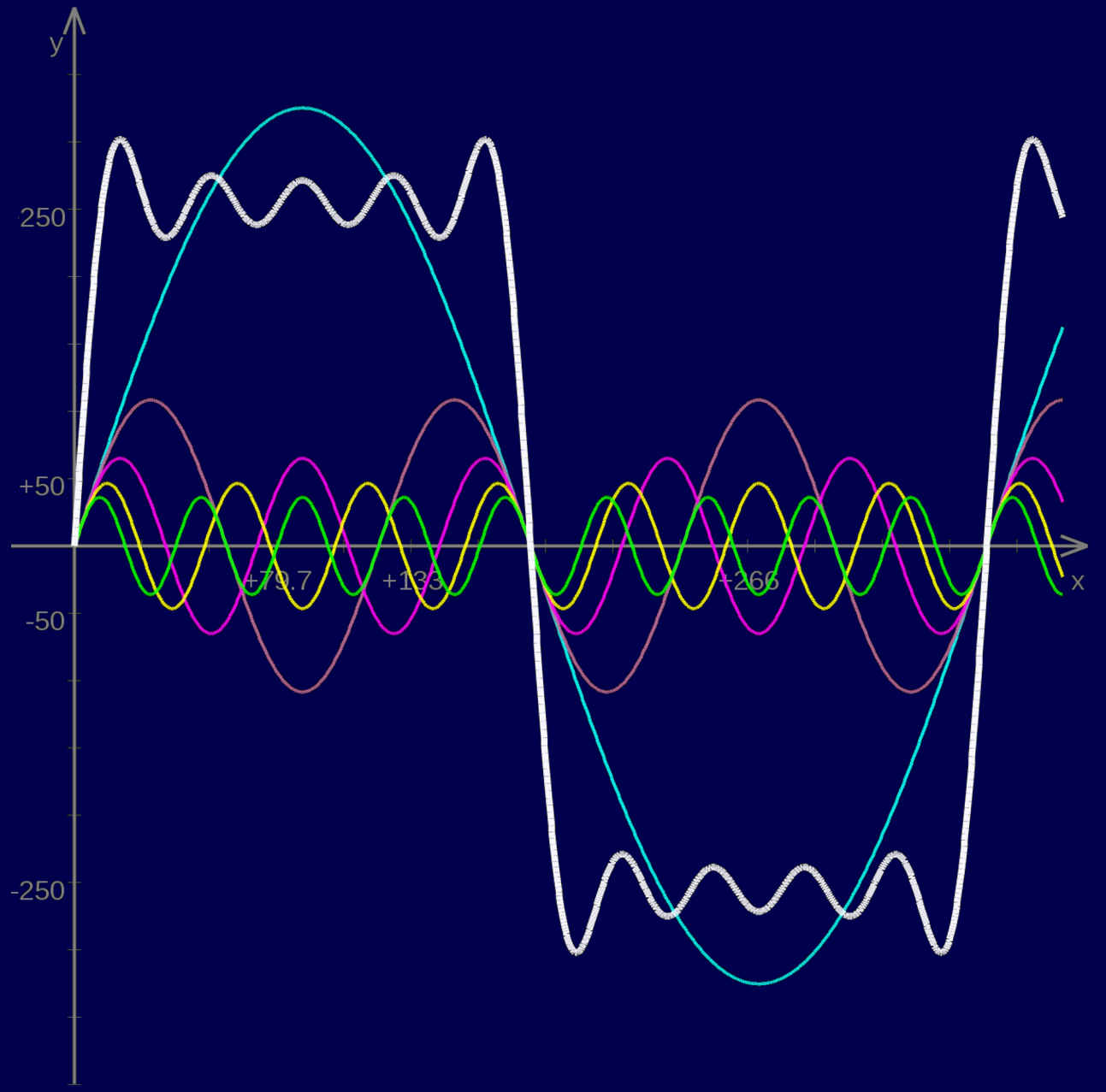
GRESHAM COLLEGE

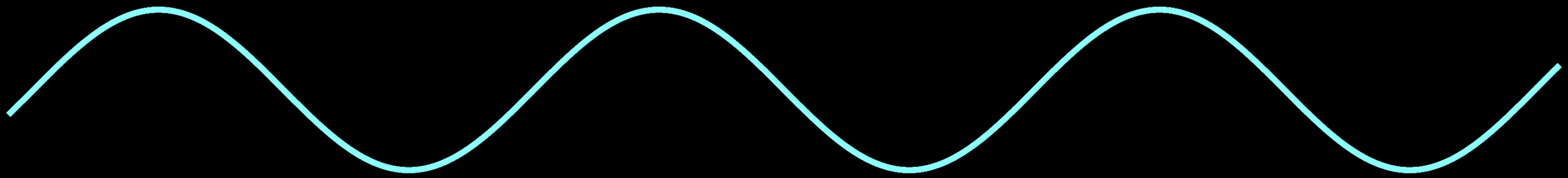
The incredible sine wave and its uses

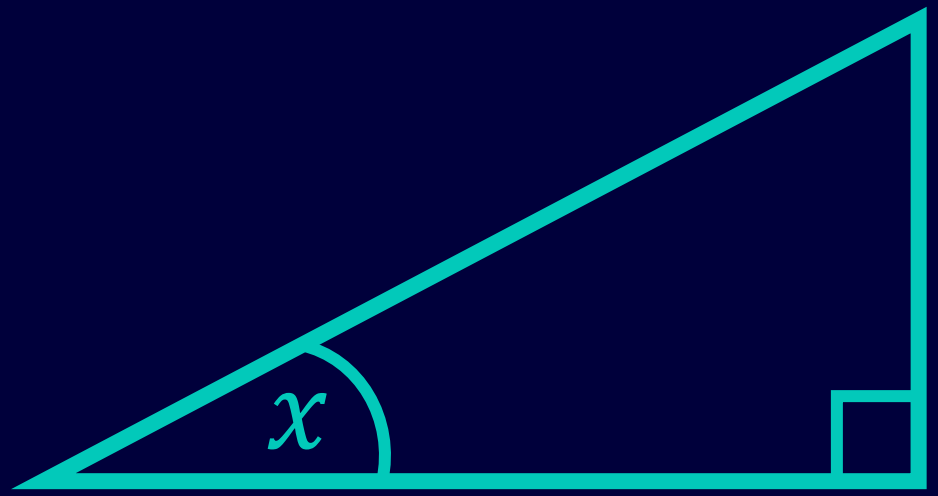
Professor Sarah Hart

Gresham Professor of Geometry

Birkbeck UNIVERSITY OF LONDON







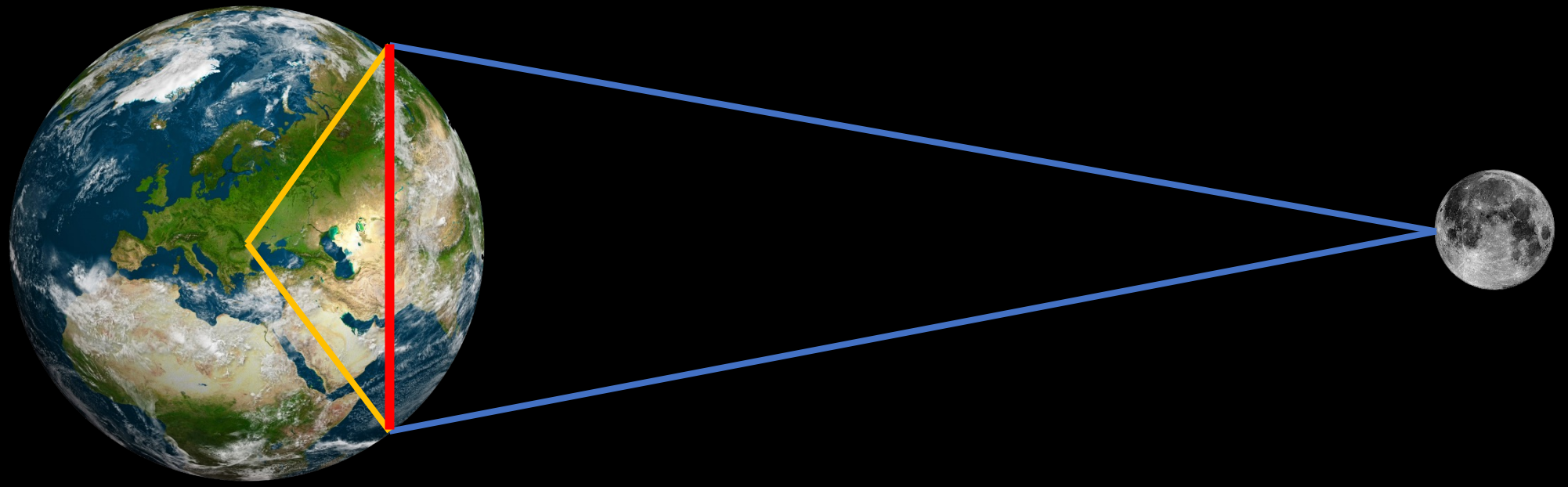
$$\sin(x) = \frac{\text{opposite}}{\text{hypotenuse}}$$

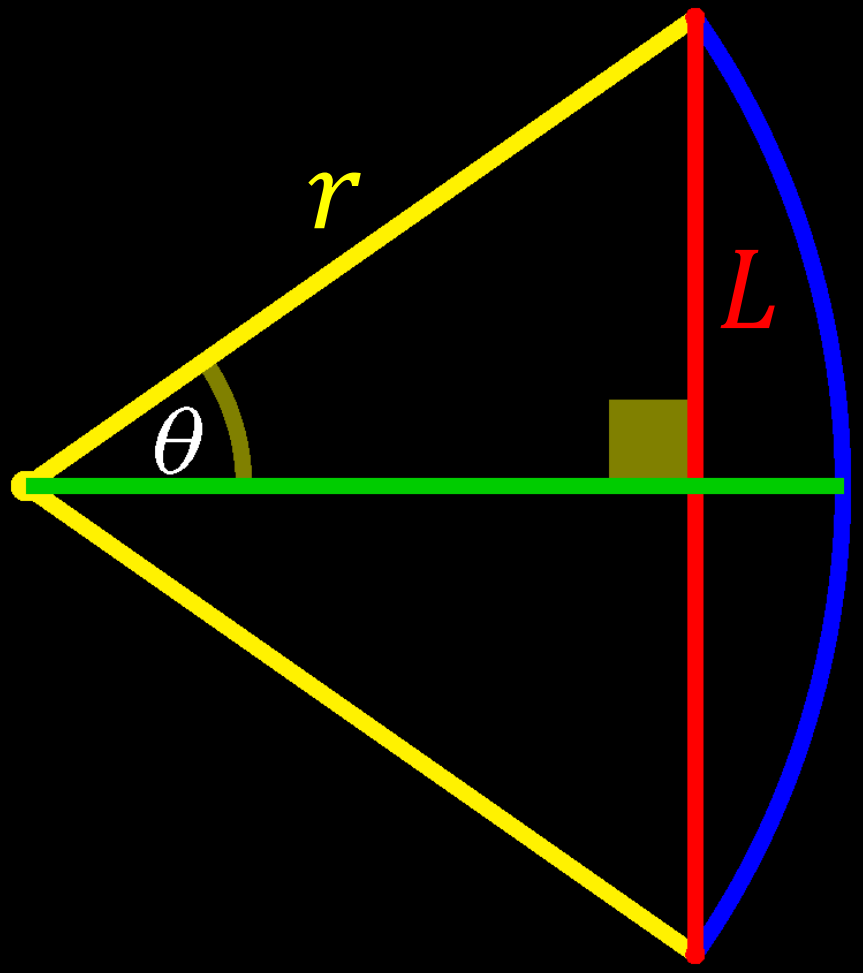
$$\cos(x) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

Hipparchus – Father of Trigonometry?

- Astronomical calculations using chords
- Calculated distance to moon at 59-67 Earth radii





- Half chord more convenient
- $\sin(\theta) = \frac{L}{r}$
- chord length depends on θ and r
- Sine depends just on angle

Sine

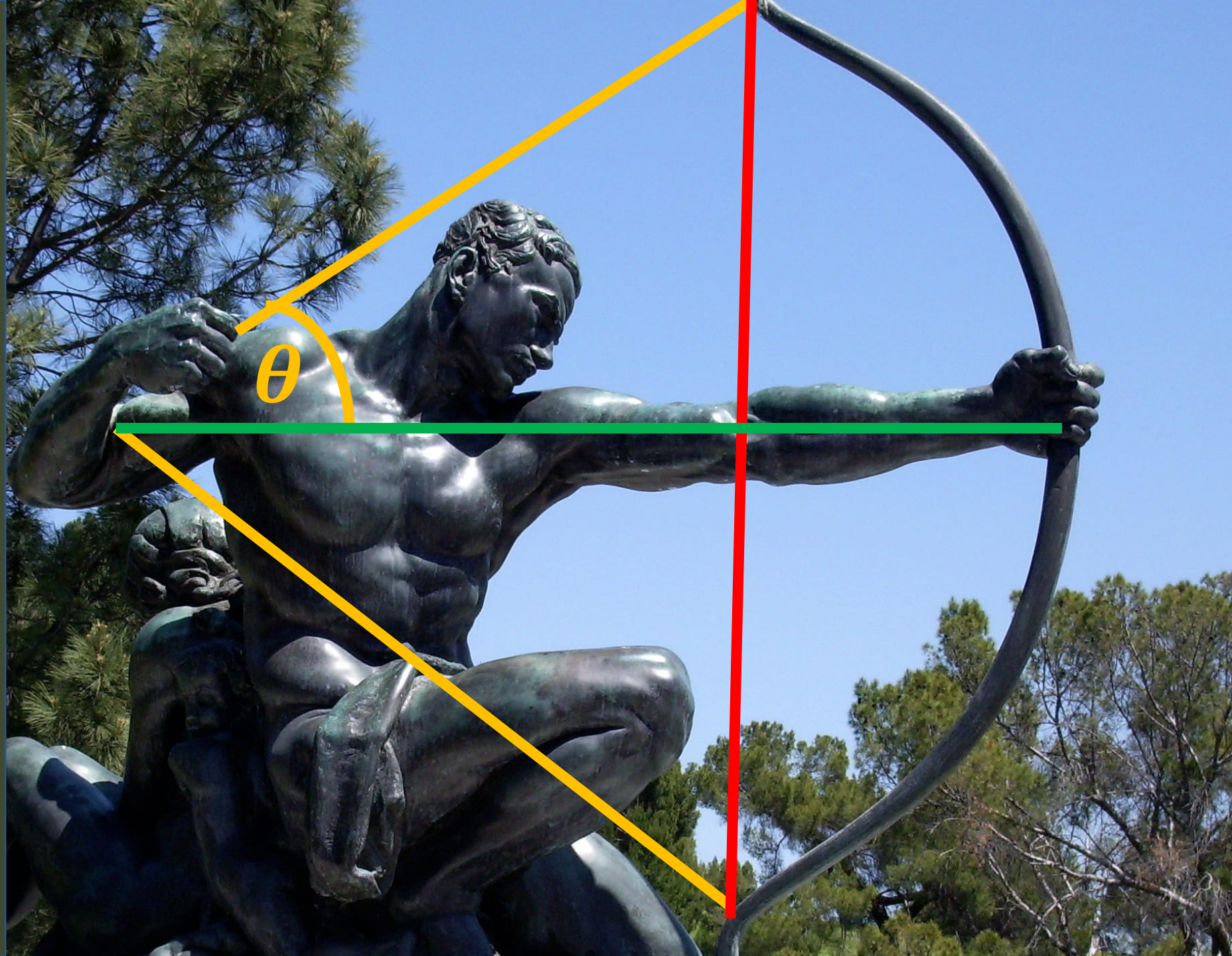
- Sanskrit

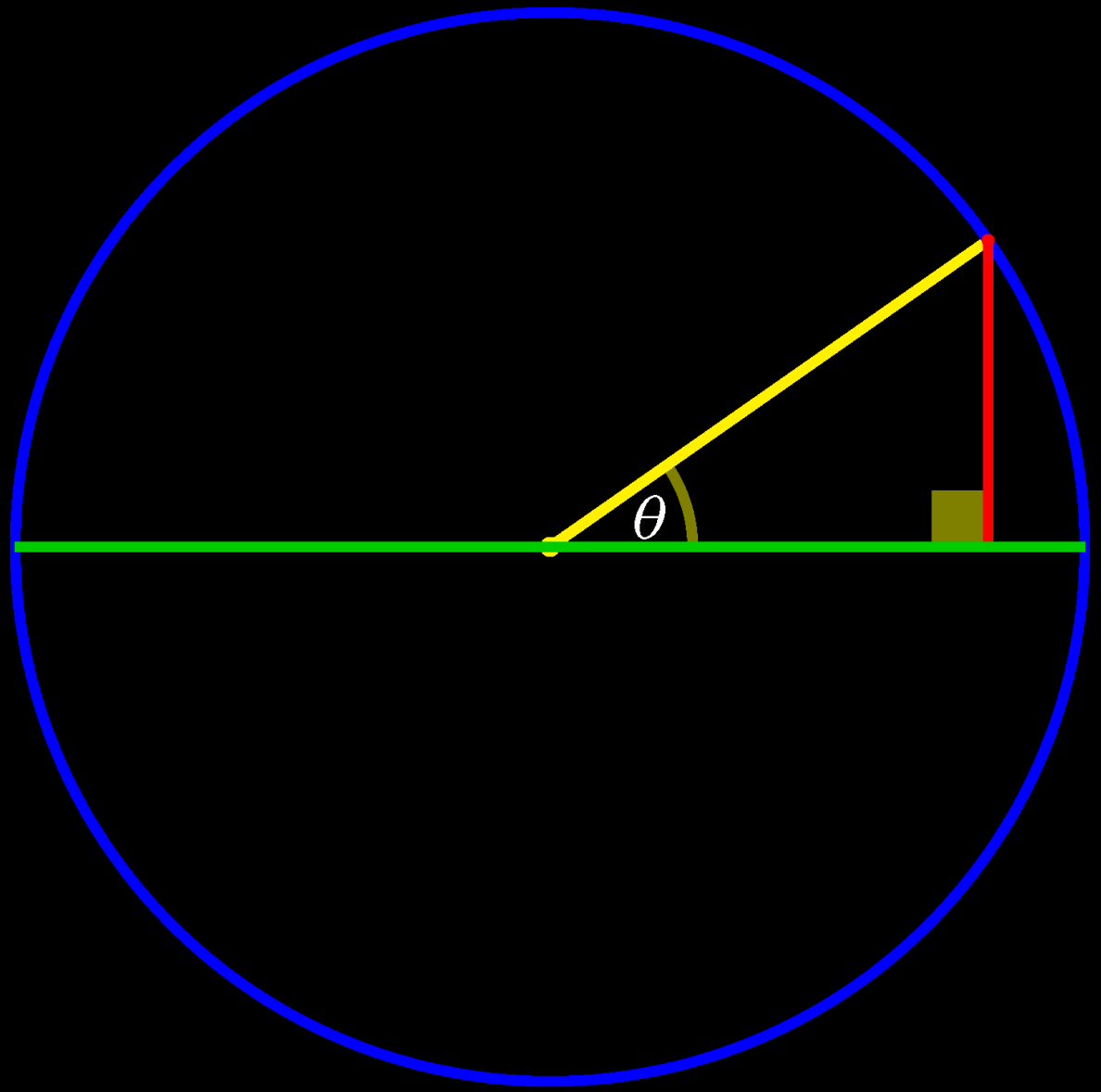
  jīvā = bowstring.

- Arabic *jība*

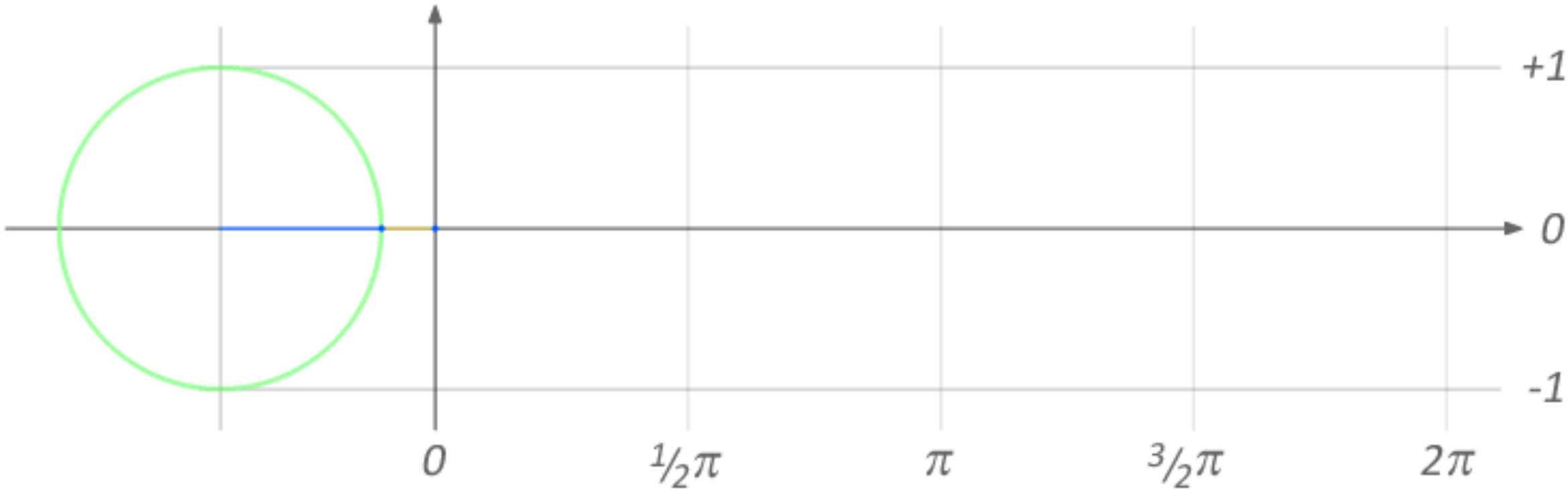
- → *jayb* = cavity

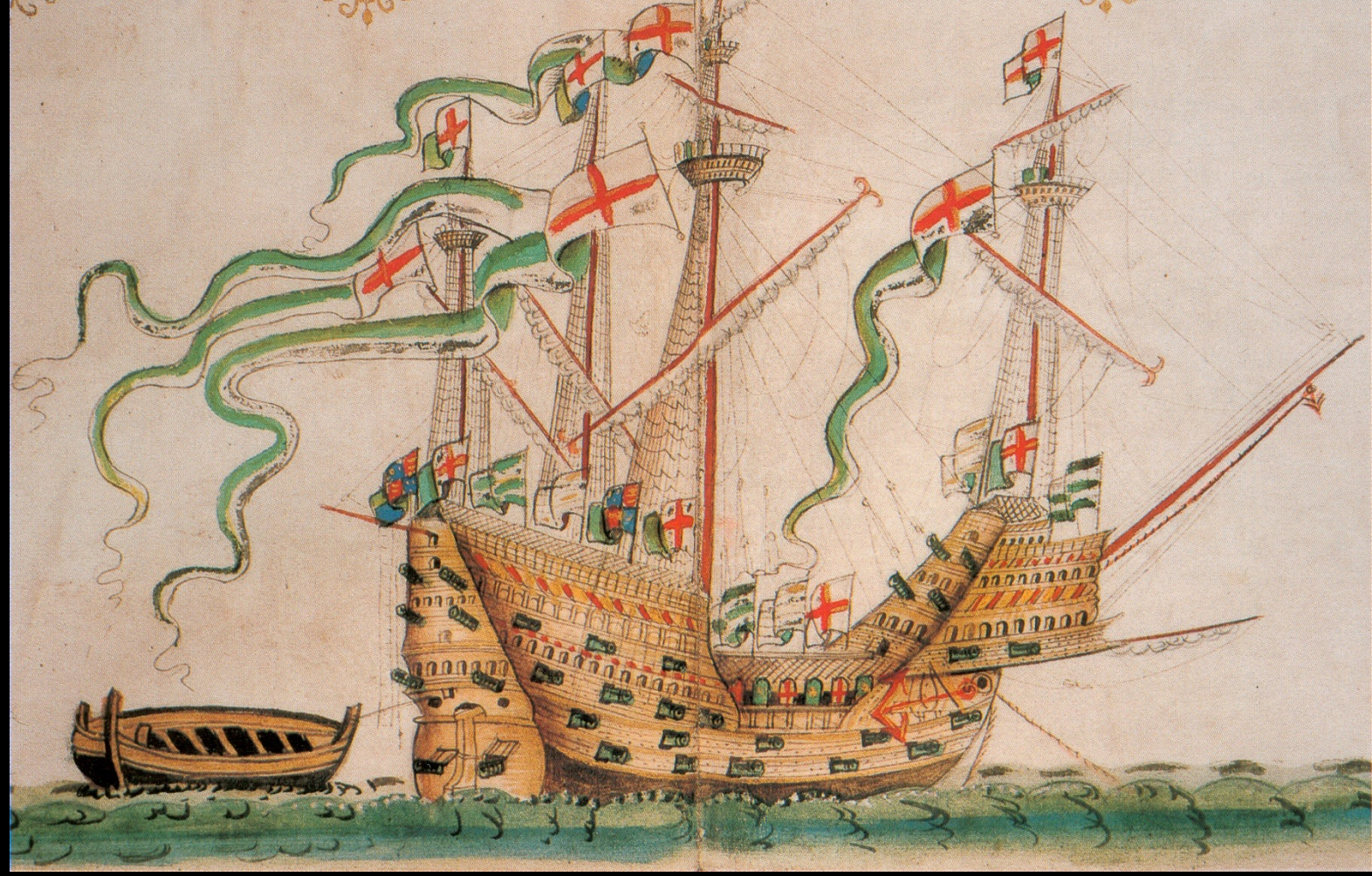
- Latin *sinus*

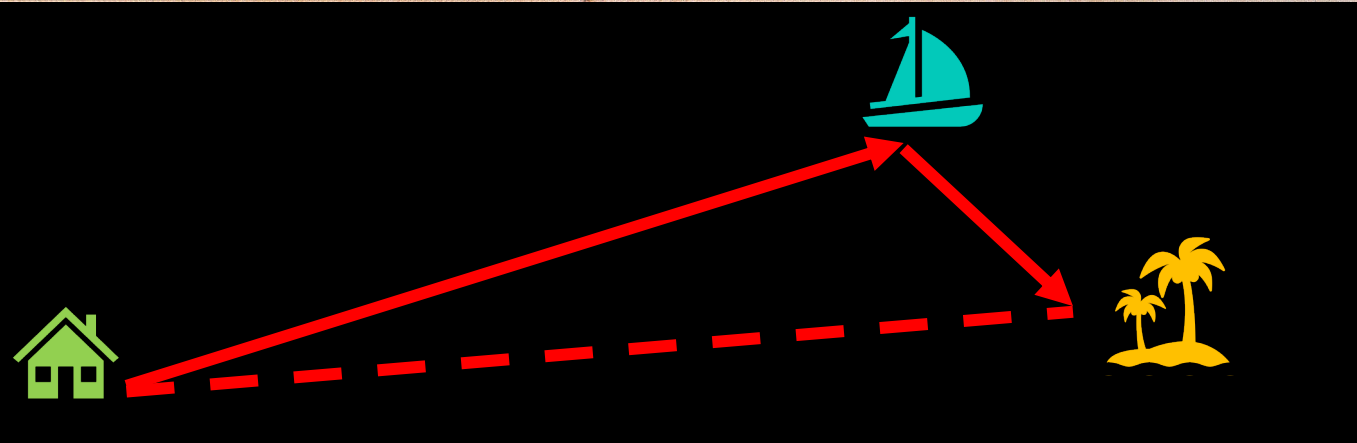


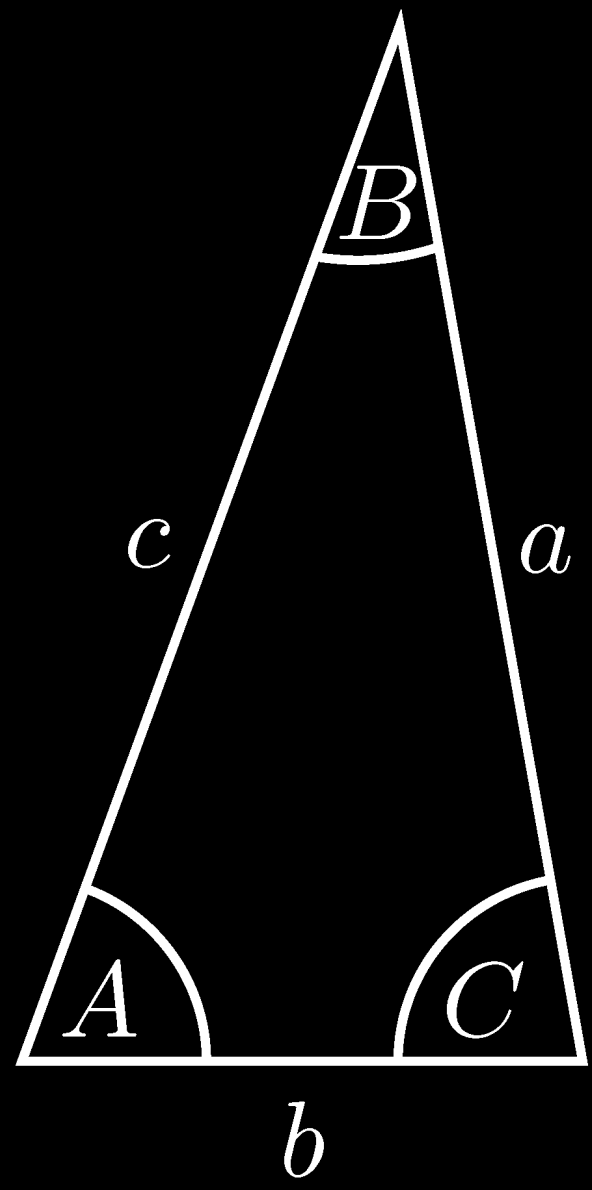


- $\sin(\theta)$ is height above horizontal of point on circumference at angle θ in circle of radius 1 (unit circle)
- 360° in circle a bit “mundane”
- Angle in radians = distance travelled round the circumference in a unit circle.


+1
0
-1
0
½π
π
3/2π
2π

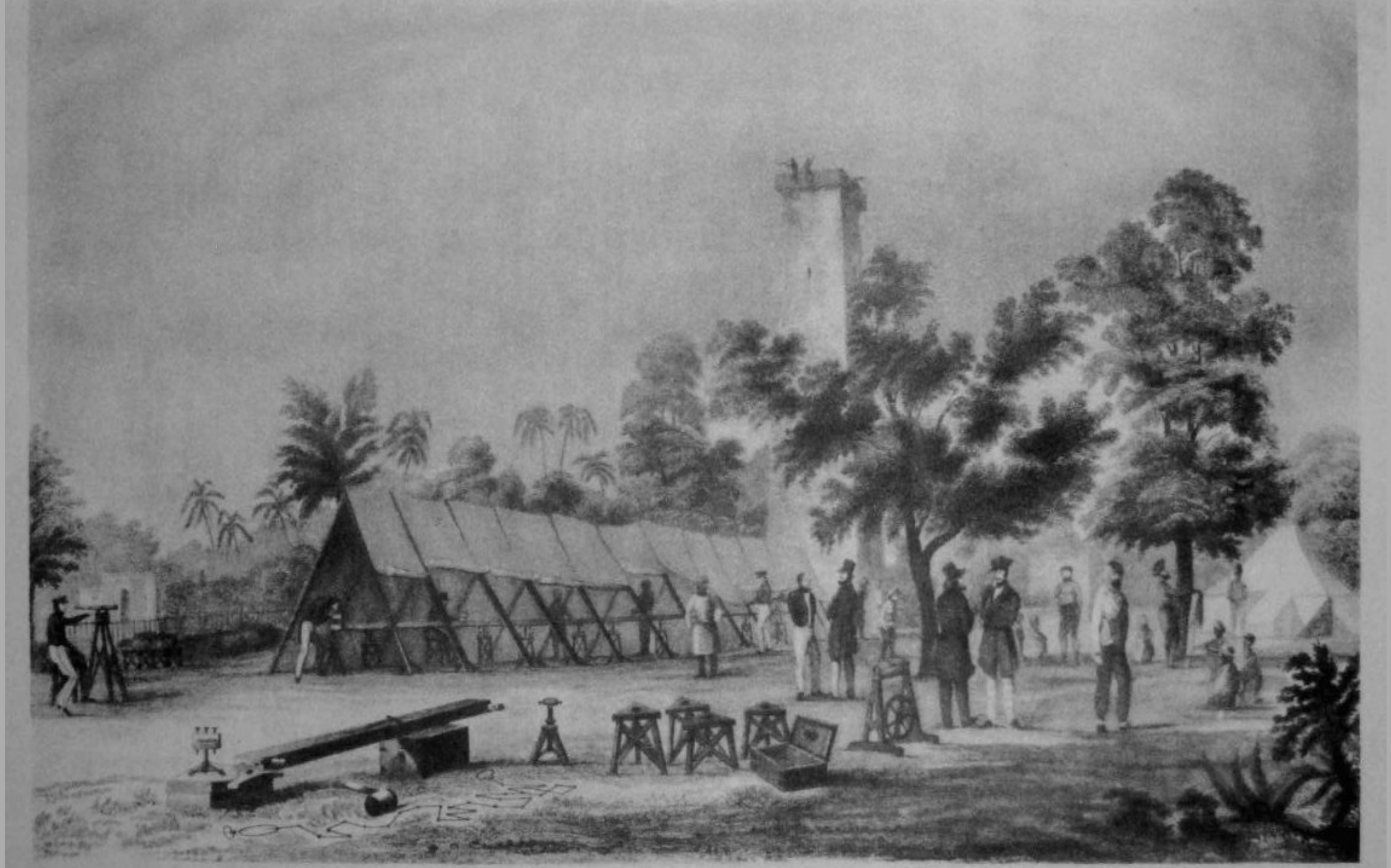






The Sine Rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



CALCUTTA BASE LINE

Regiomontanus (1436-76)

- Wrote one of first trigonometry textbooks in Europe, *De Triangulis Omnimodis*
- *"You who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles."*
- Early book of trig tables (1490)

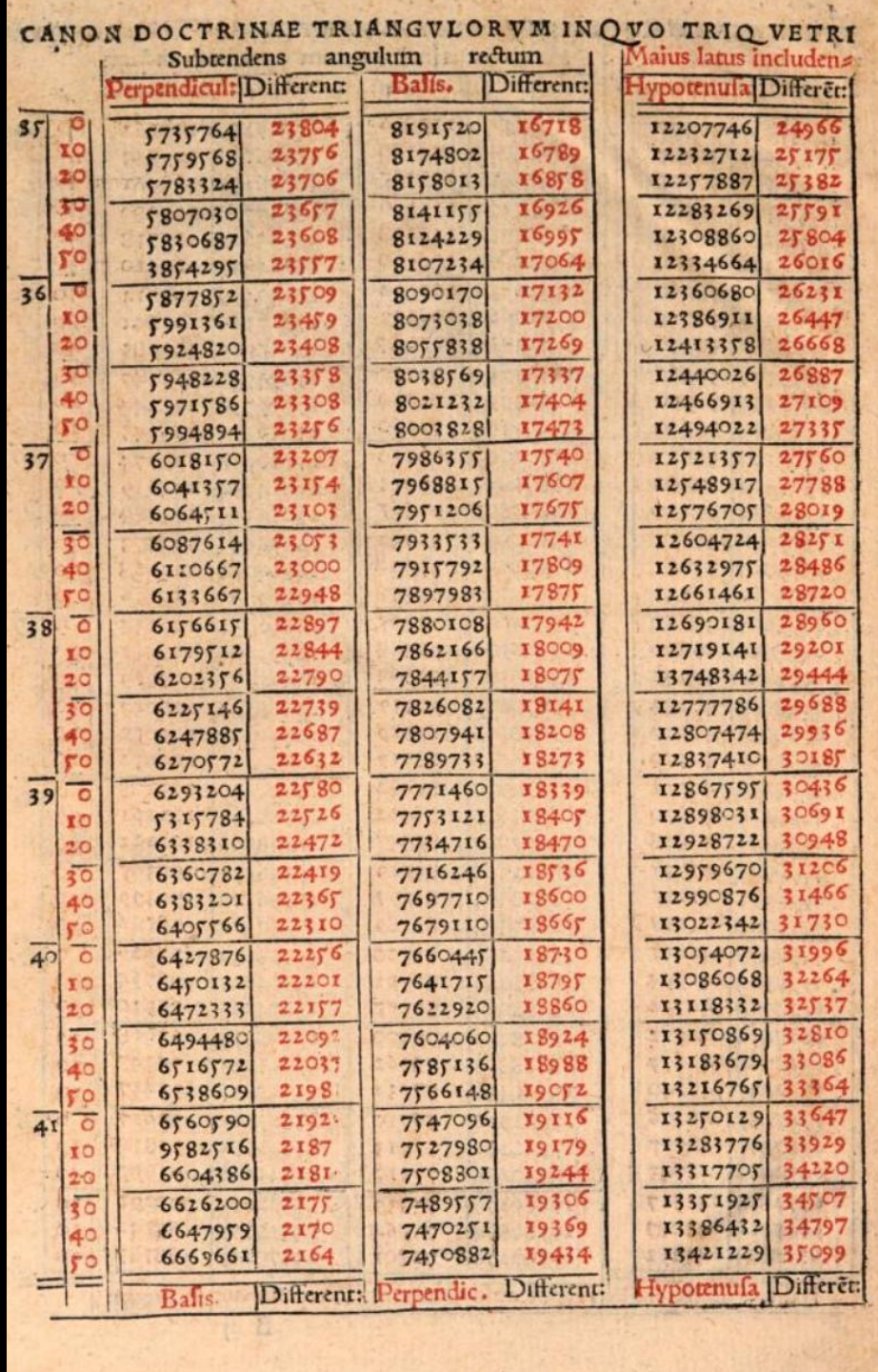
Georg Rheticus (1514-1574)

- Very accurate trig tables (1551)
  non-standard functions, and banned!

- Trig formulae developed to calculate tables:

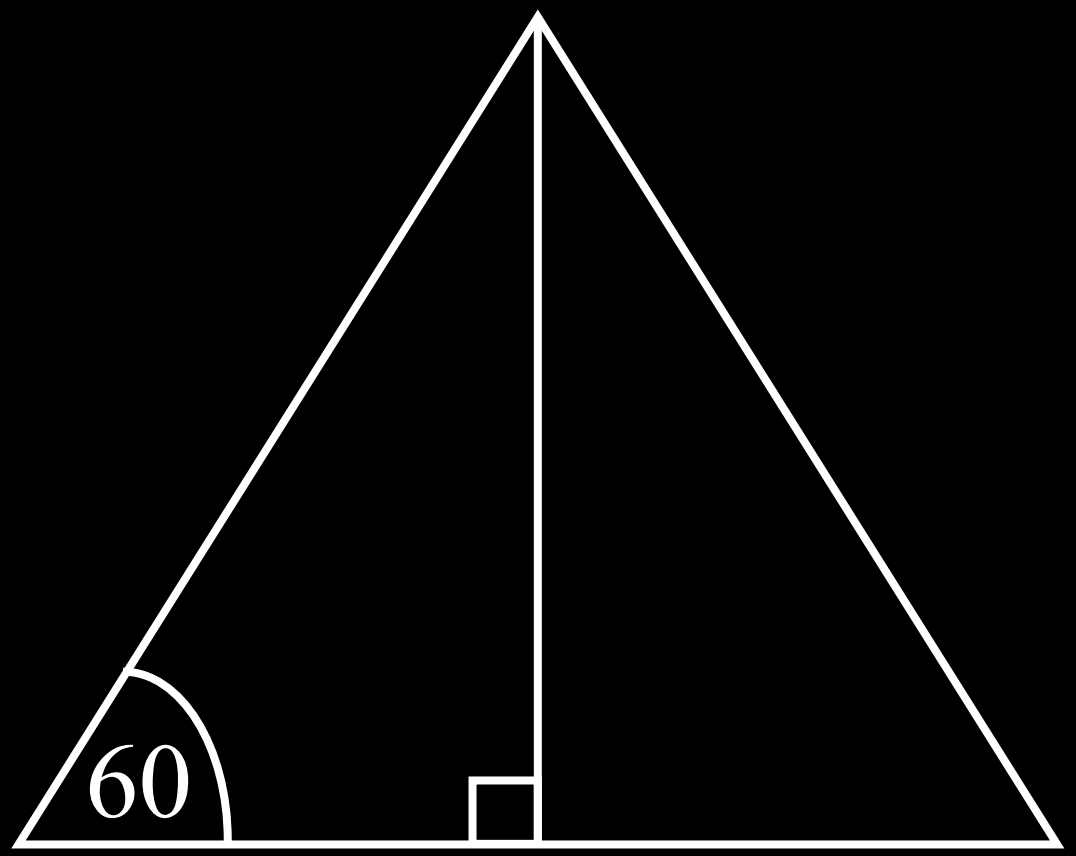
$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{2}(1-\cos x)}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$



CANON DOCTRINAE TRIANGVLORVM IN QVO TRIQ VETRI

		Subtendens angulum rectum: Perpendicul:	Differenti:	Basis.	Differenti:	Maius latus includens: Hypotenusa	Differēti:
35	0	5735764	23804	8191520	16718	12207746	24966
	10	5759568	23756	8174802	16789	12232712	25175
	20	5783324	23706	8158013	16858	12257887	25382
	30	5807030	23657	8141155	16926	12283269	25591
	40	5830687	23608	8124229	16995	12308860	25804
	50	5854295	23557	8107234	17064	12334664	26016
36	0	5877852	23509	8090170	17132	12360680	26231
	10	5901361	23459	8073038	17200	12386911	26447
	20	5924820	23408	8055838	17269	12413358	26668
	30	5948228	23358	8038569	17337	12440026	26887
	40	5971586	23308	8021232	17404	12466913	27109
	50	5994894	23256	8003828	17473	12494022	27335
37	0	6018150	23207	7986355	17540	12521357	27560
	10	6041357	23154	7968815	17607	12548917	27788
	20	6064511	23103	7951206	17675	12576705	28019
	30	6087614	23053	7933533	17741	12604724	28251
	40	6110667	23000	7915792	17809	12632975	28486
	50	6133667	22948	7897983	17875	12661461	28720
38	0	6156615	22897	7880108	17942	12690181	28960
	10	6179512	22844	7862166	18009	12719141	29201
	20	6202356	22790	7844157	18075	13748342	29444
	30	6225146	22739	7826082	18141	12777786	29688
	40	6247885	22687	7807941	18208	12807474	29936
	50	6270572	22632	7789733	18273	12837410	30185
39	0	6293204	22580	7771460	18339	12867595	30436
	10	6315784	22526	7753121	18405	12898031	30691
	20	6338310	22472	7734716	18470	12928722	30948
	30	6360782	22419	7716246	18536	12959670	31206
	40	6383201	22365	7697710	18600	12990876	31466
	50	6405566	22310	7679110	18665	13022342	31730
40	0	6427876	22256	7660445	18730	13054072	31996
	10	6450132	22201	7641715	18795	13086068	32264
	20	6472333	22157	7622920	18860	13118332	32537
	30	6494480	22092	7604060	18924	13150869	32810
	40	6516572	22037	7585136	18988	13183679	33086
	50	6538609	2198	7566148	19052	13216765	33364
41	0	6560590	2192	7547096	19116	13250129	33647
	10	9582516	2187	7527980	19179	13283776	33929
	20	6604386	2181	7508801	19244	13317705	34220
	30	6626200	2175	7489557	19306	13351925	34507
	40	6647959	2170	7470251	19369	13386432	34797
	50	6669661	2164	7450882	19434	13421229	35099
		Basis.	Differenti:	Perpendic.	Differenti:	Hypotenusa	Differēti: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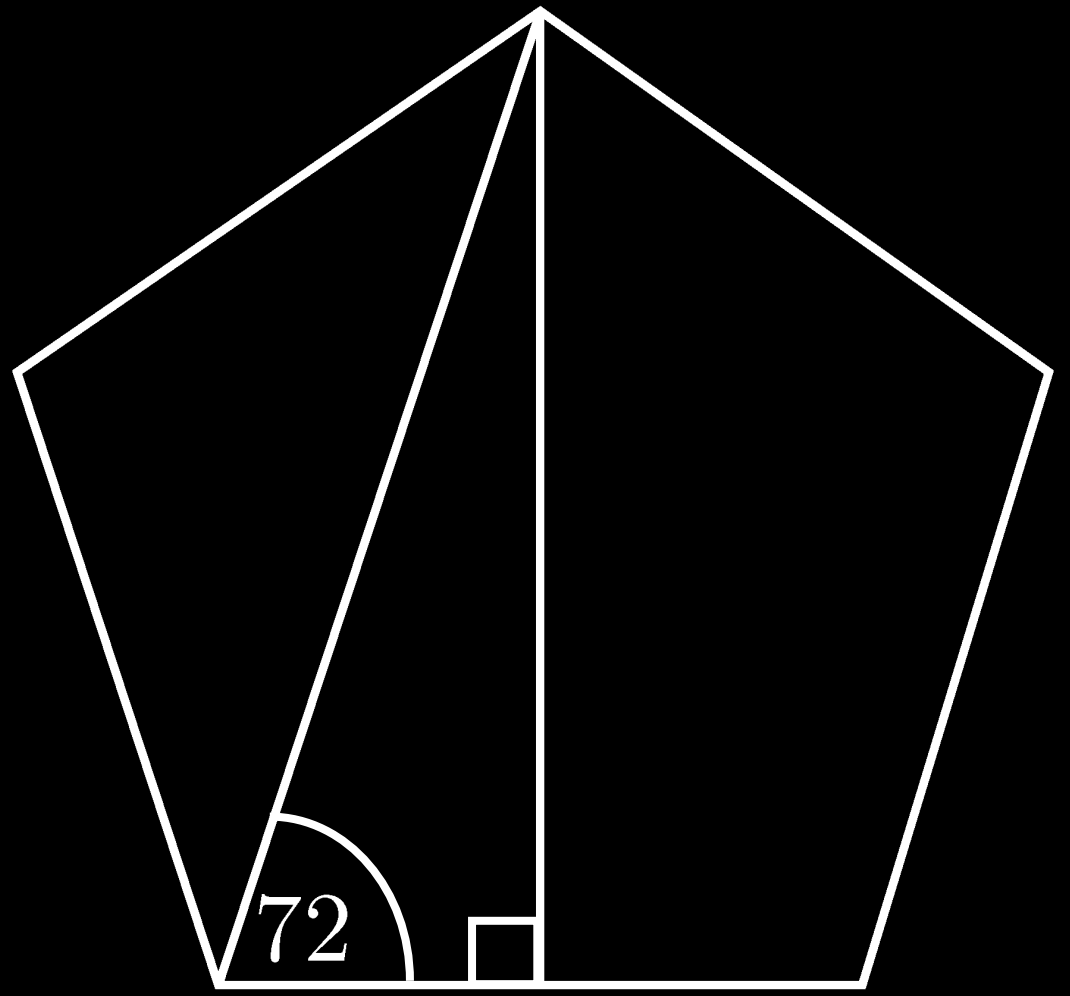
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Prosthaphaeresis

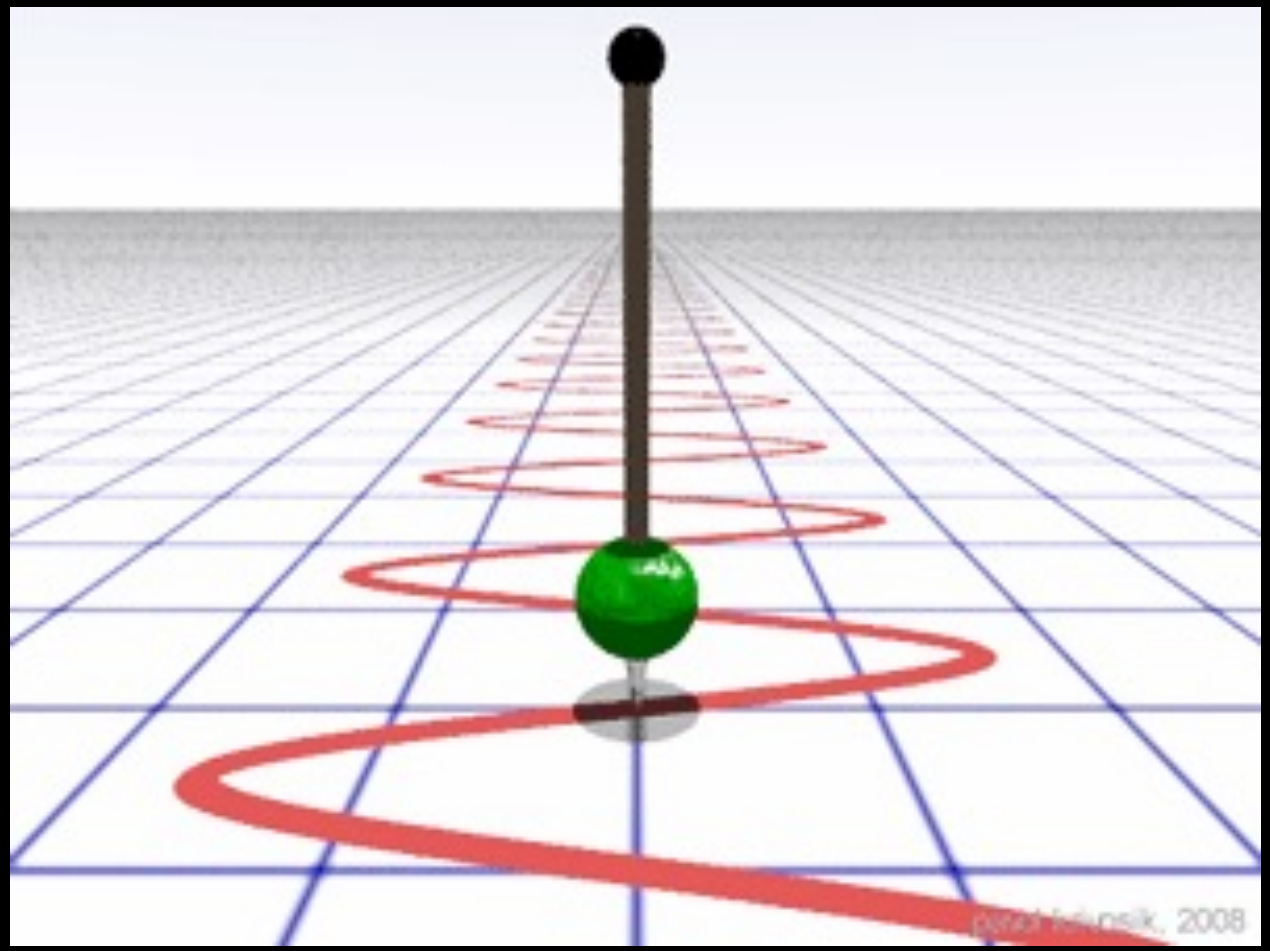
$$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

To multiply two large numbers

Toy example: 123×456

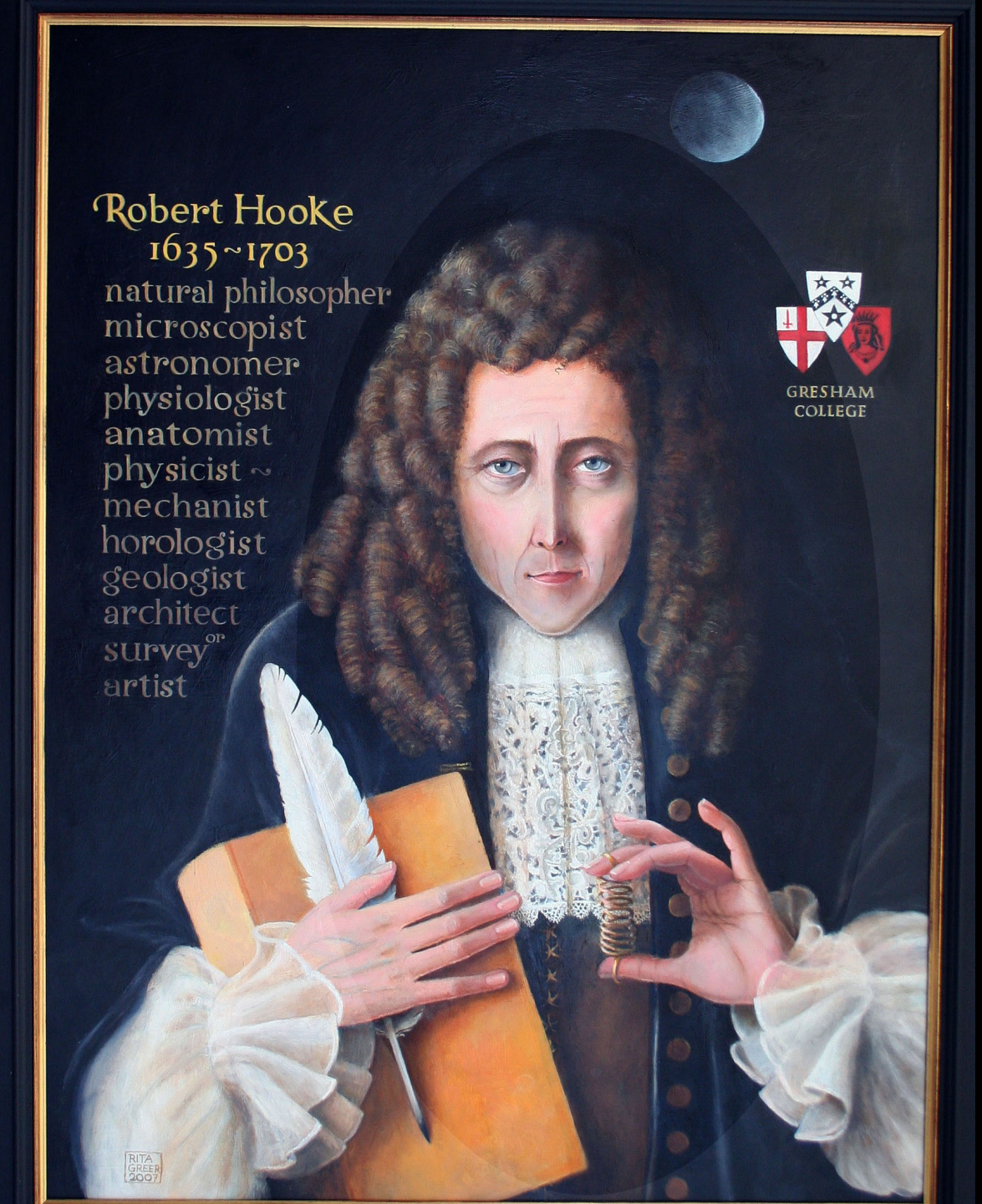
- Scale by power of 10 to get $0 \le A \le B \le 1$ → $A = 0.123, B = 0.456$
- Find x, y so that $\cos x = A$ and $\cos y = B$ → $x = 82.9347, y = 62.8707$
- Find $x + y$ and $x - y$ → $\cos(x+y) = -0.82713,$
- Find their cosines $\cos(x-y) = 0.93931$
- Take the average → Average = 0.056088
- Scale back to find the required product → $123 \times 456 = 56{,}088.$

Oscillation and sine waves

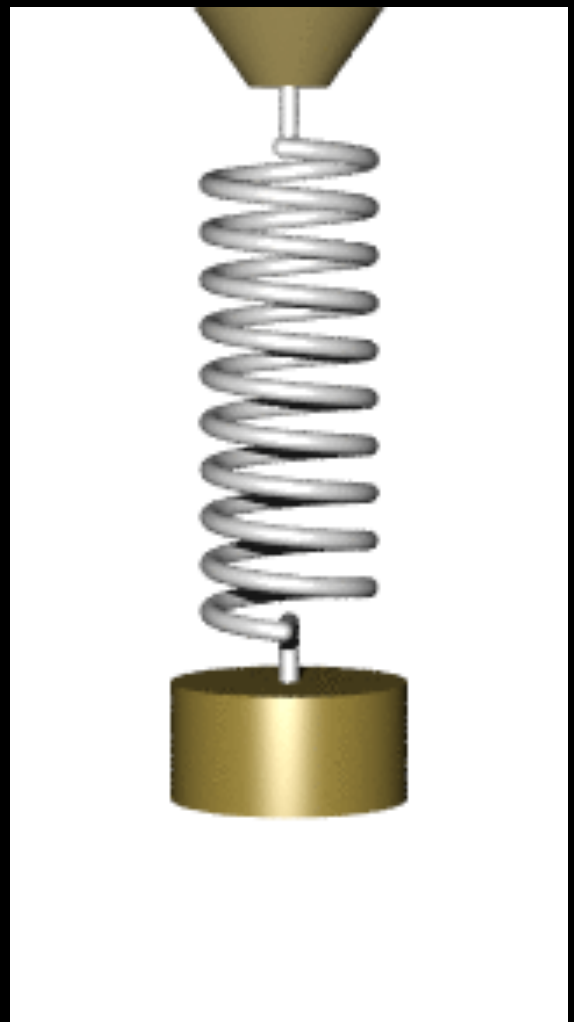


Sine function models systems where:

- force acts towards equilibrium position;
- force is proportional to distance from that position.

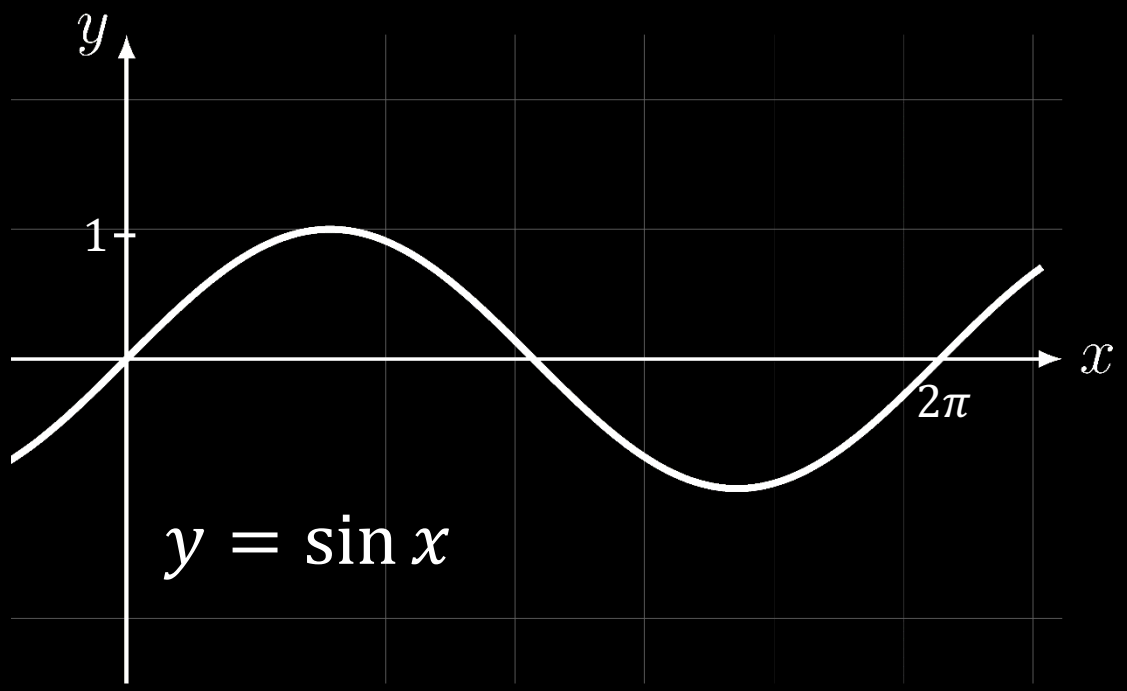


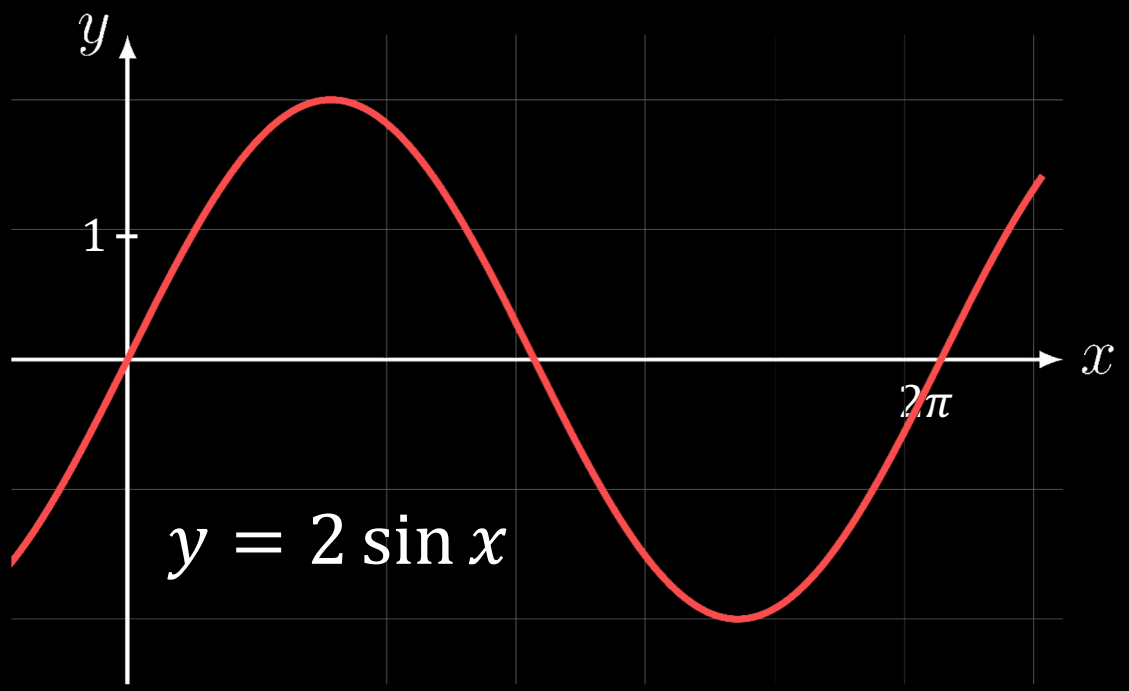
- Hooke's Law (1676): *ceiiinosssttuv*
- "*ut tensio, sic vis*"

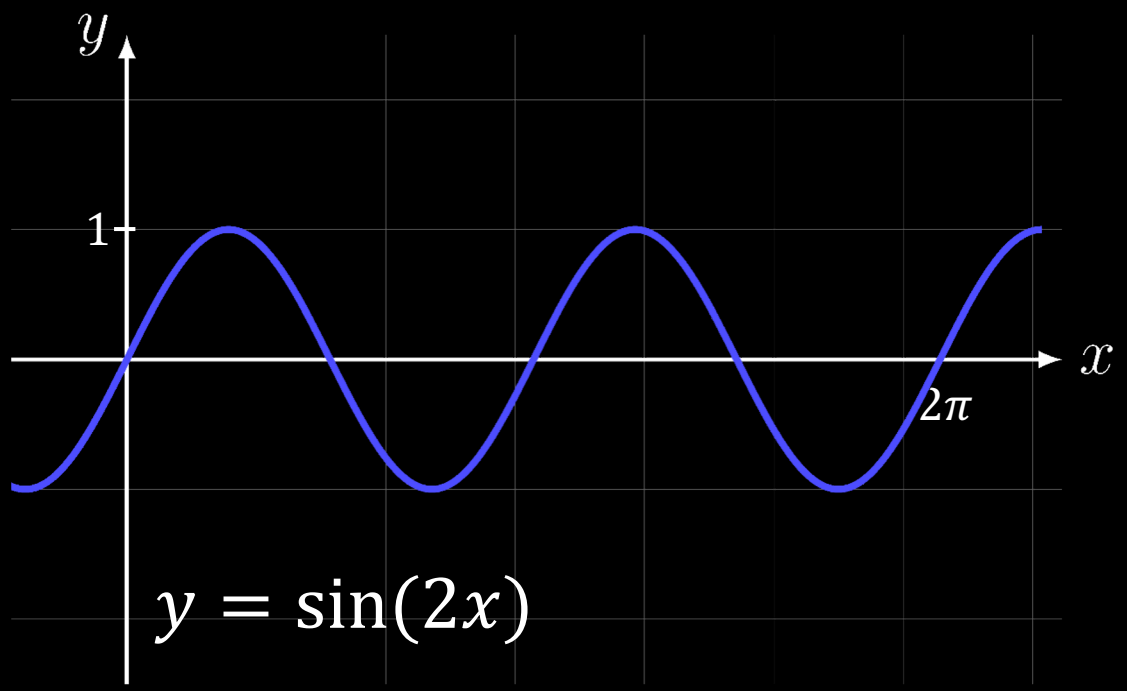


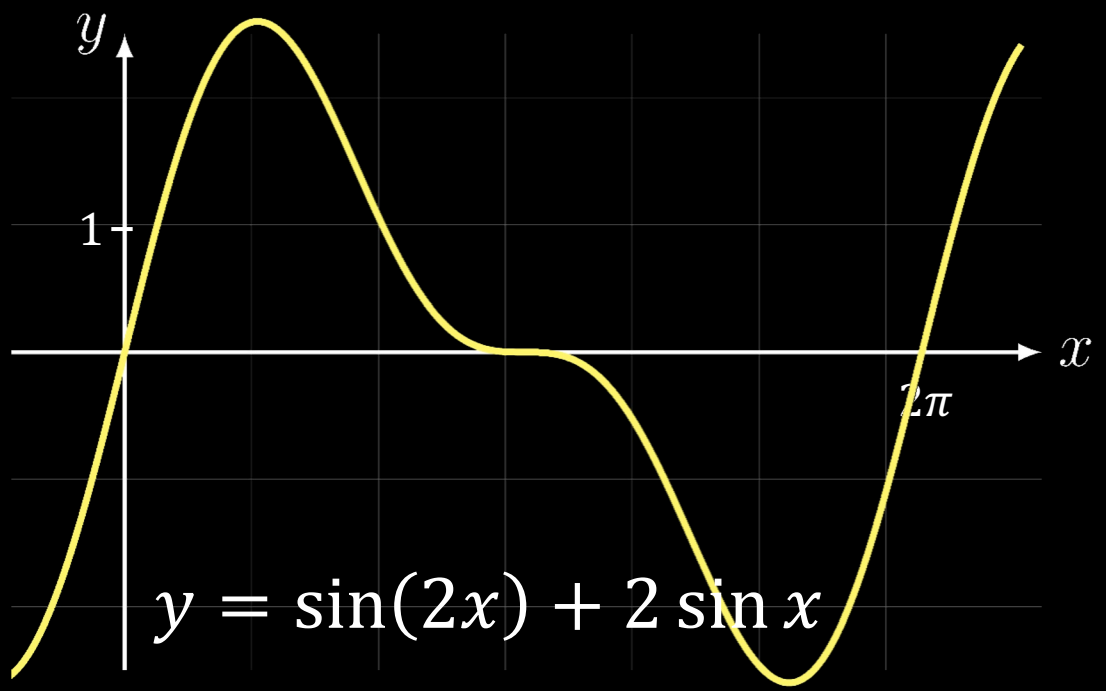
Joseph Fourier (1768-1830)

- *Théorie analytique de la chaleur* (1822), mathematical analysis of heat flow.


y
1
x
2π
y = sin x


y
1
x
2π
y = 2 sin x

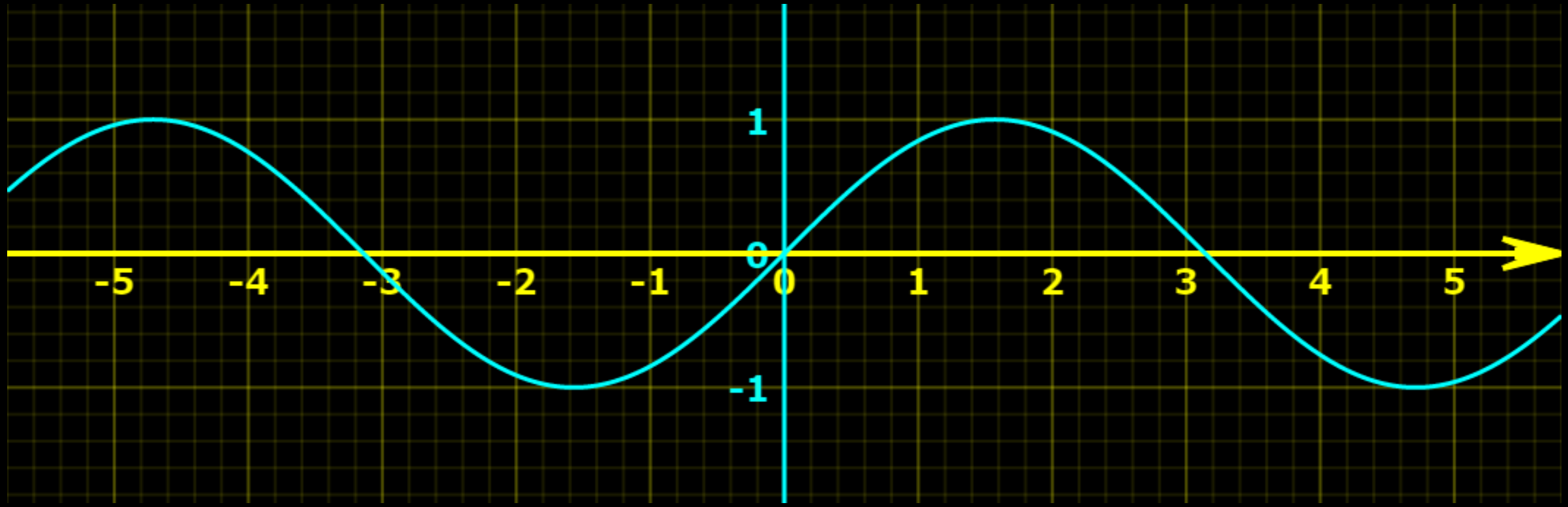

y
1
x
2π
y = sin(2x)


y
1
x
2π
y = sin(2x) + 2 sin x

Joseph Fourier (1768-1830)

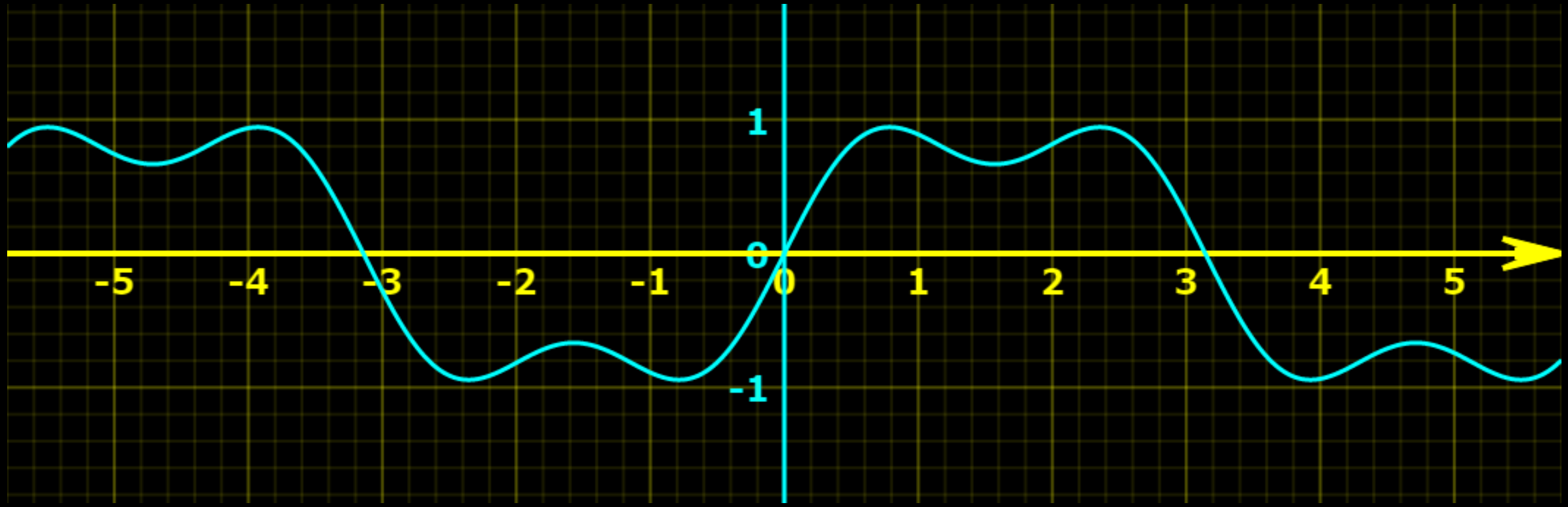
- *Théorie analytique de la chaleur* (1822), mathematical analysis of heat flow.
- Showed that any* periodic function is made up of sine waves – AND we know how to do it!

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



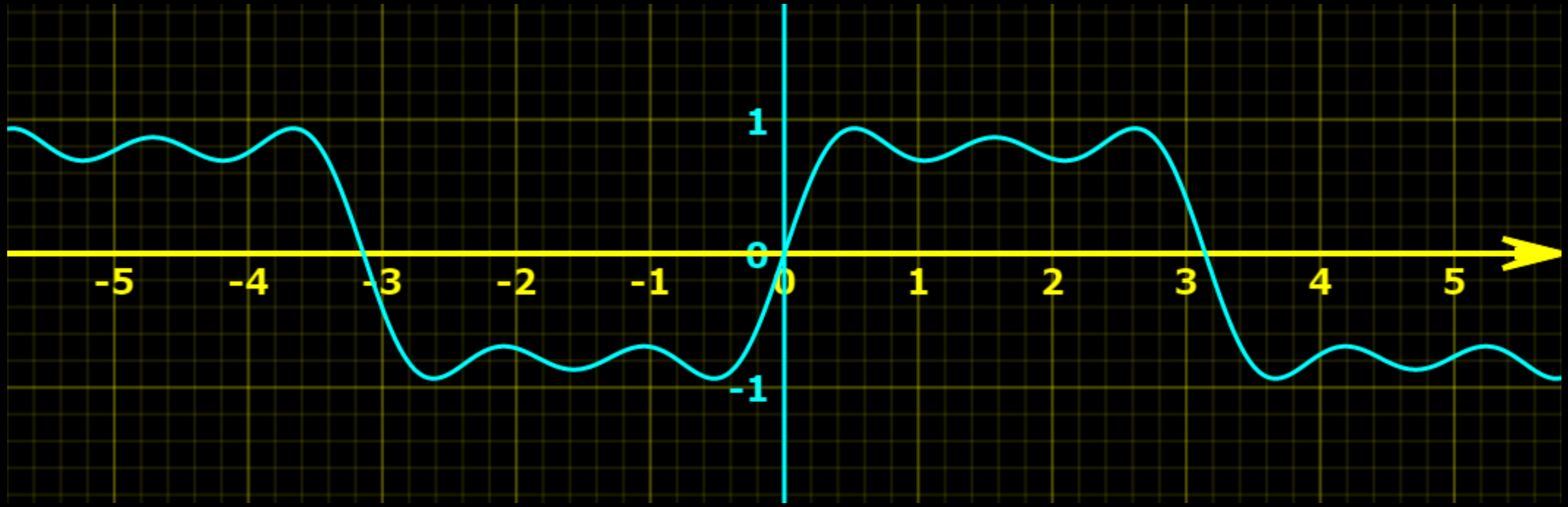
1 term of the series

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



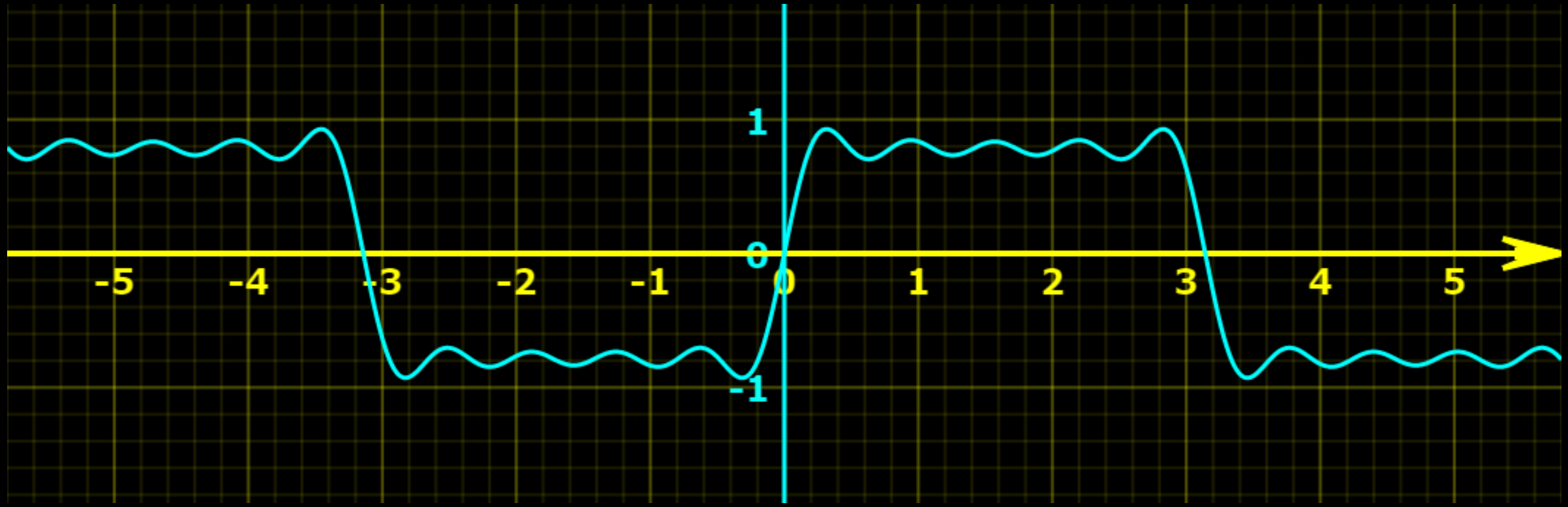
2 terms of the series

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



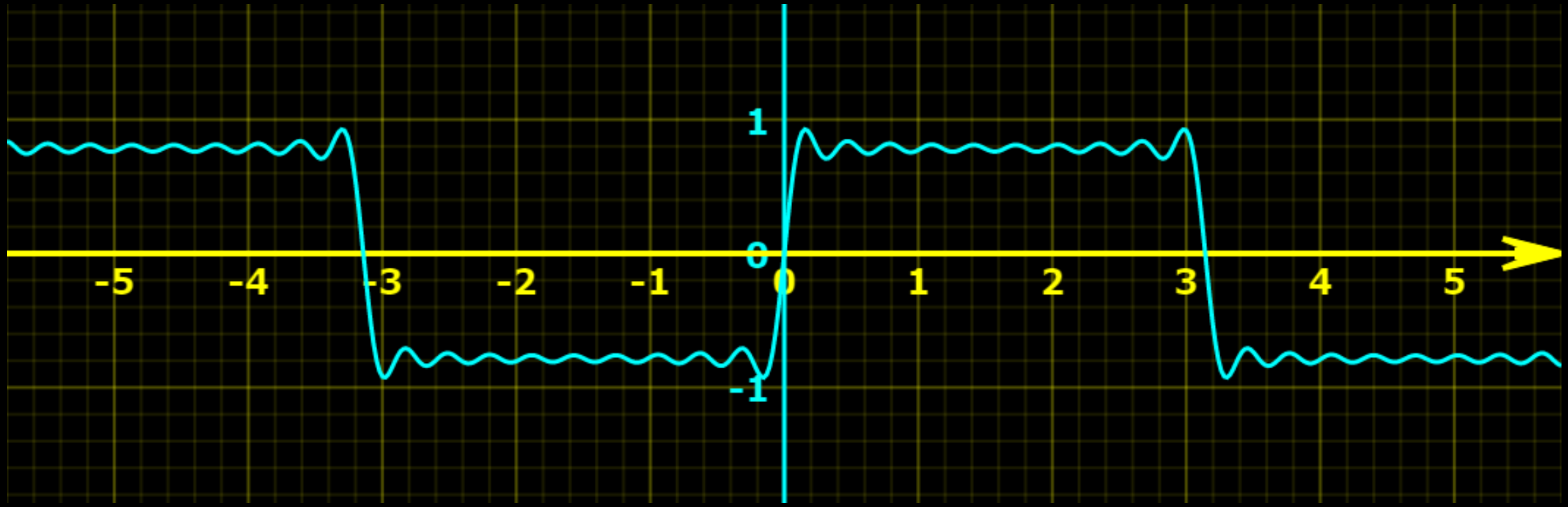
3 terms of the series

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



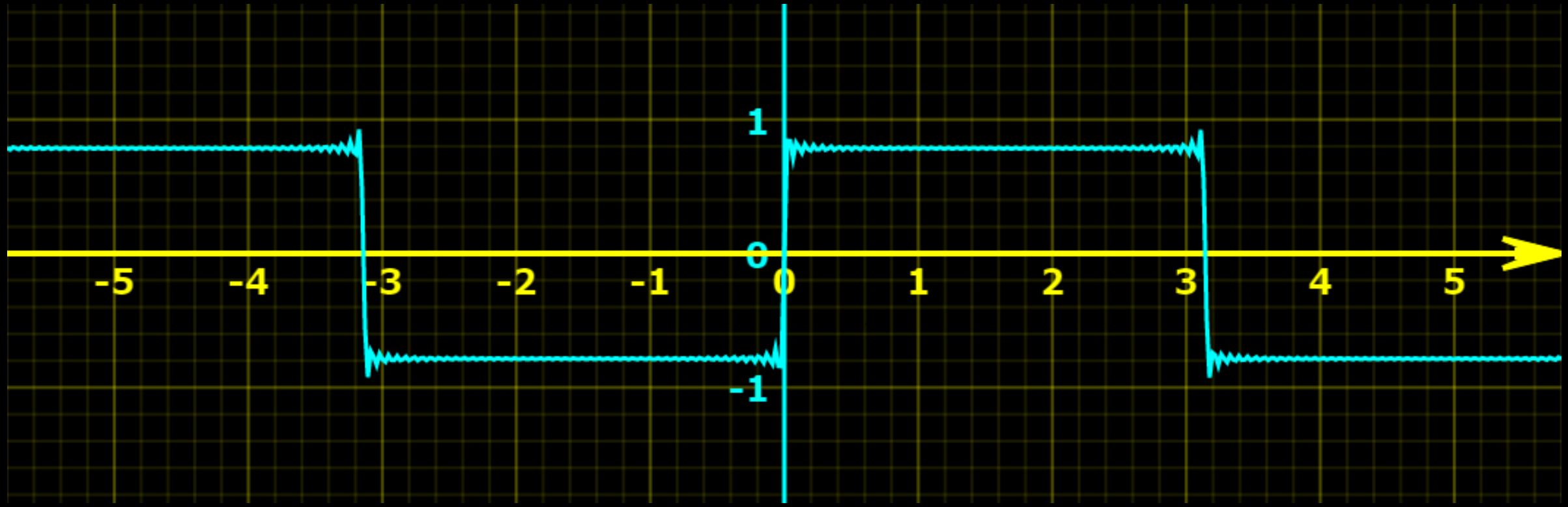
5 terms

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



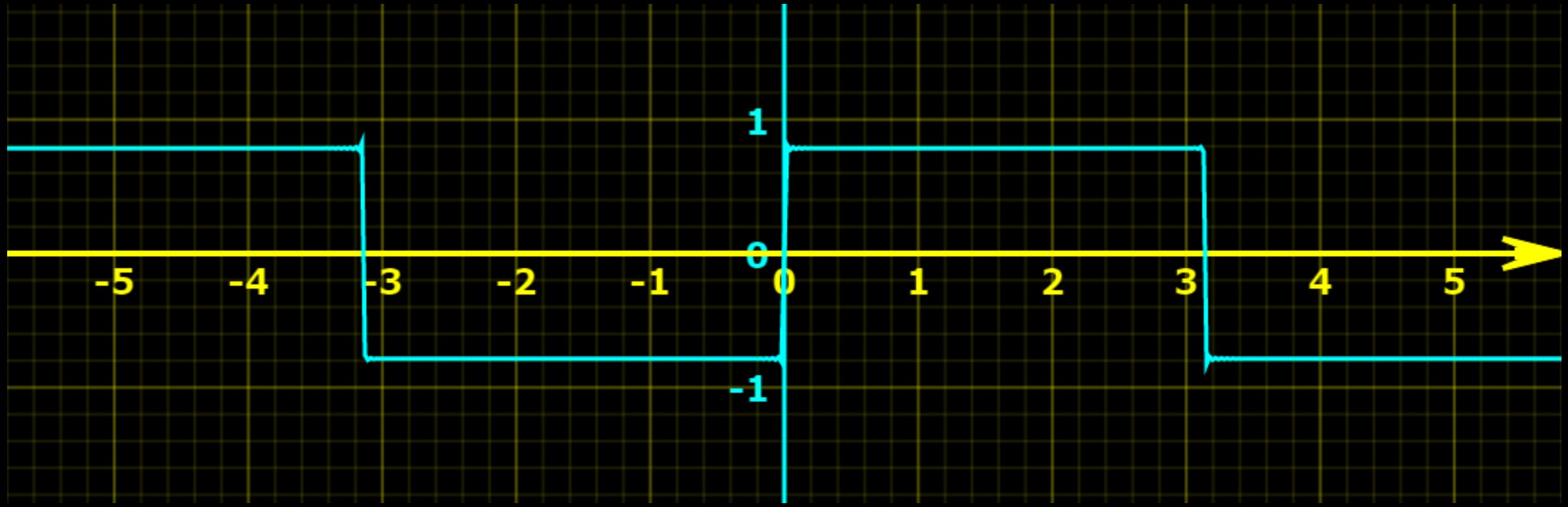
10 terms

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



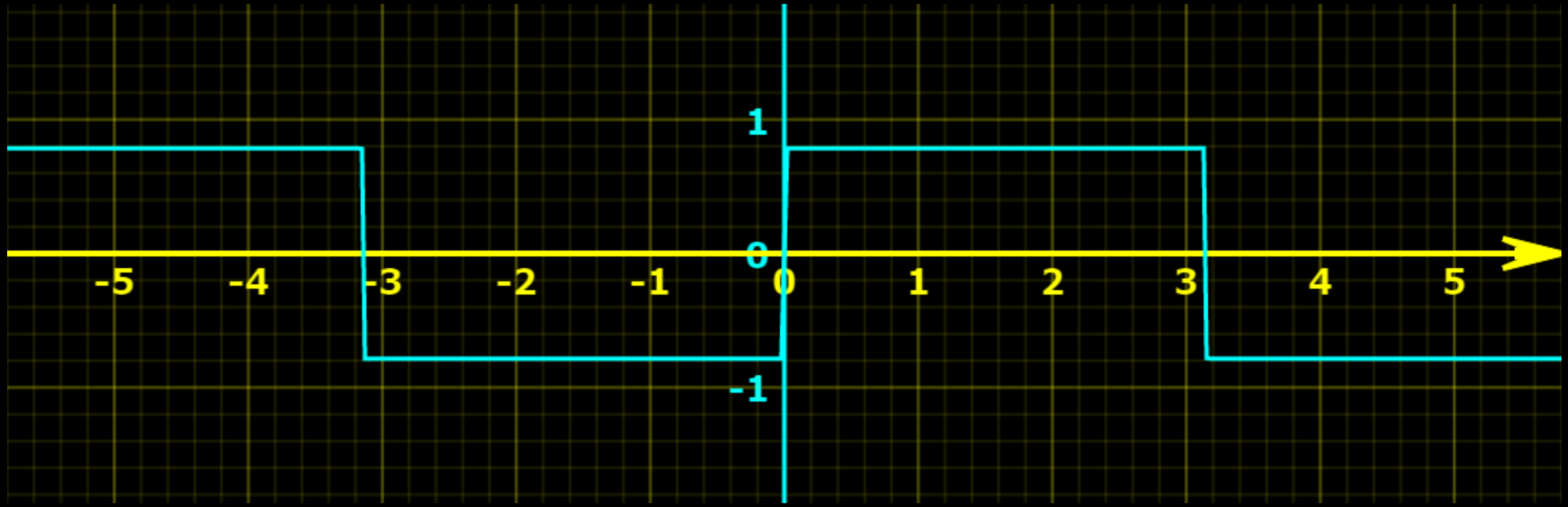
50 terms

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



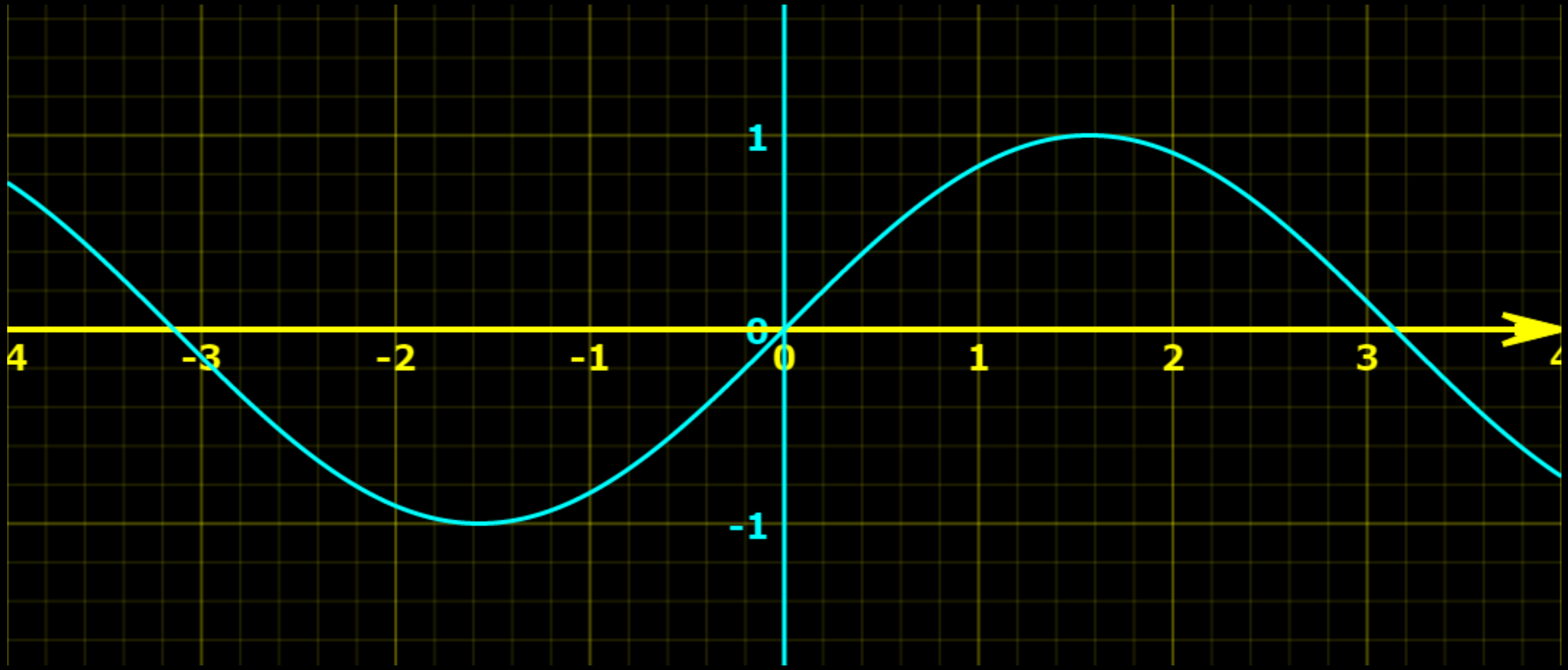
500 terms

What is $\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$?



5000 terms

What is $\sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x) + \cdots$?



1 term

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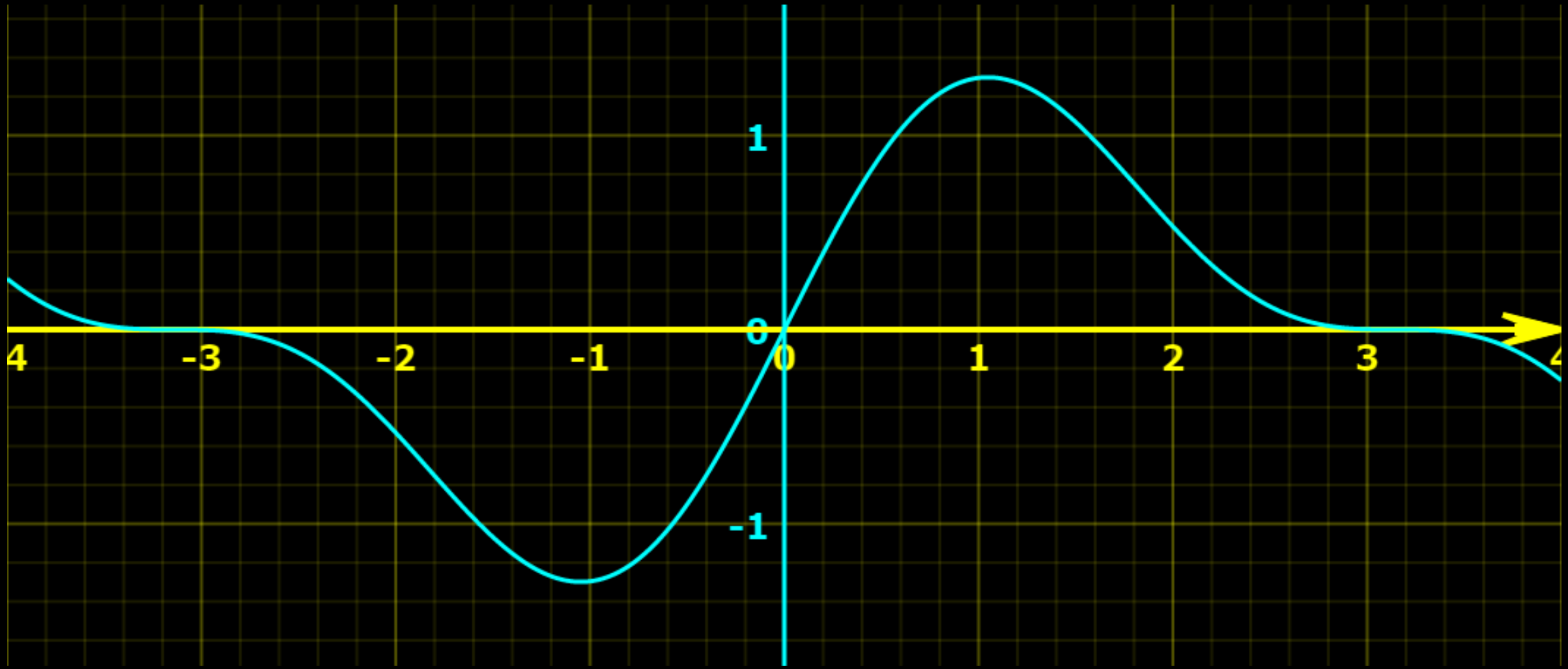

2 terms

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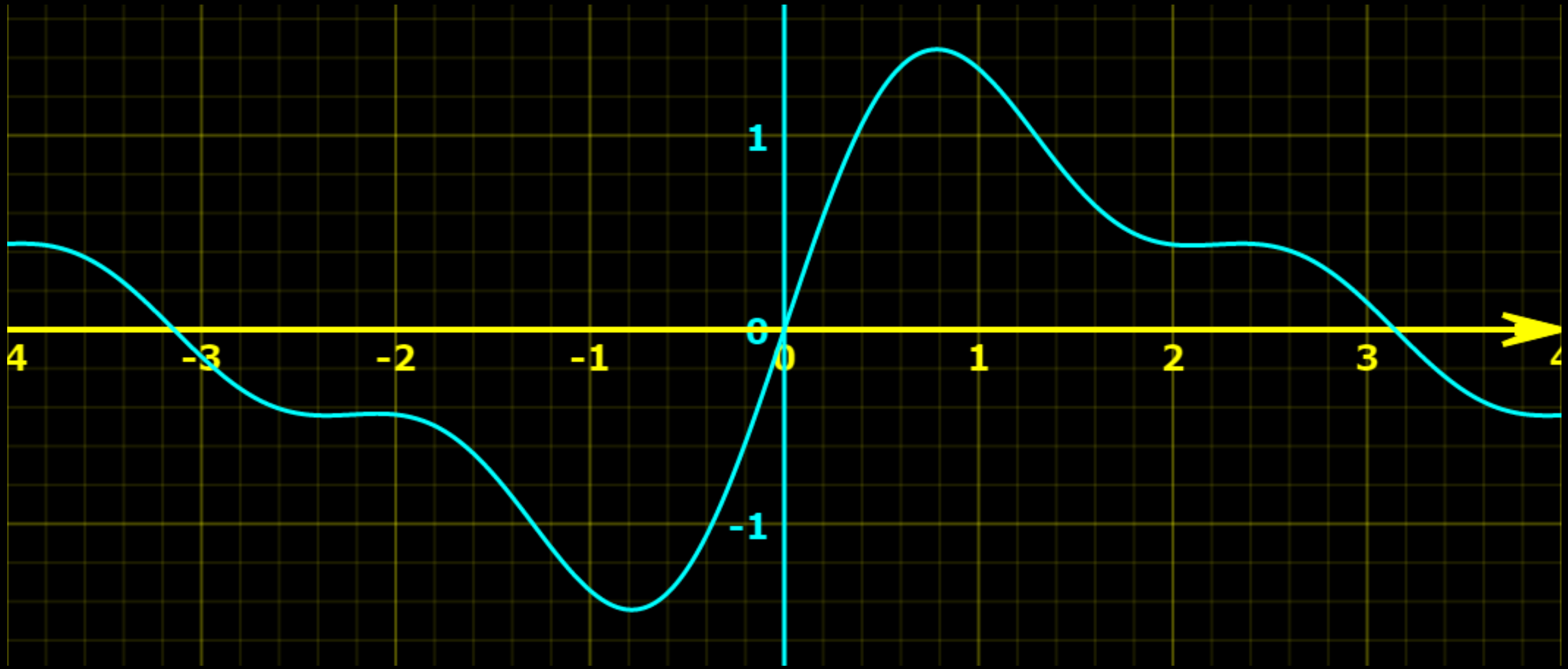

3 terms

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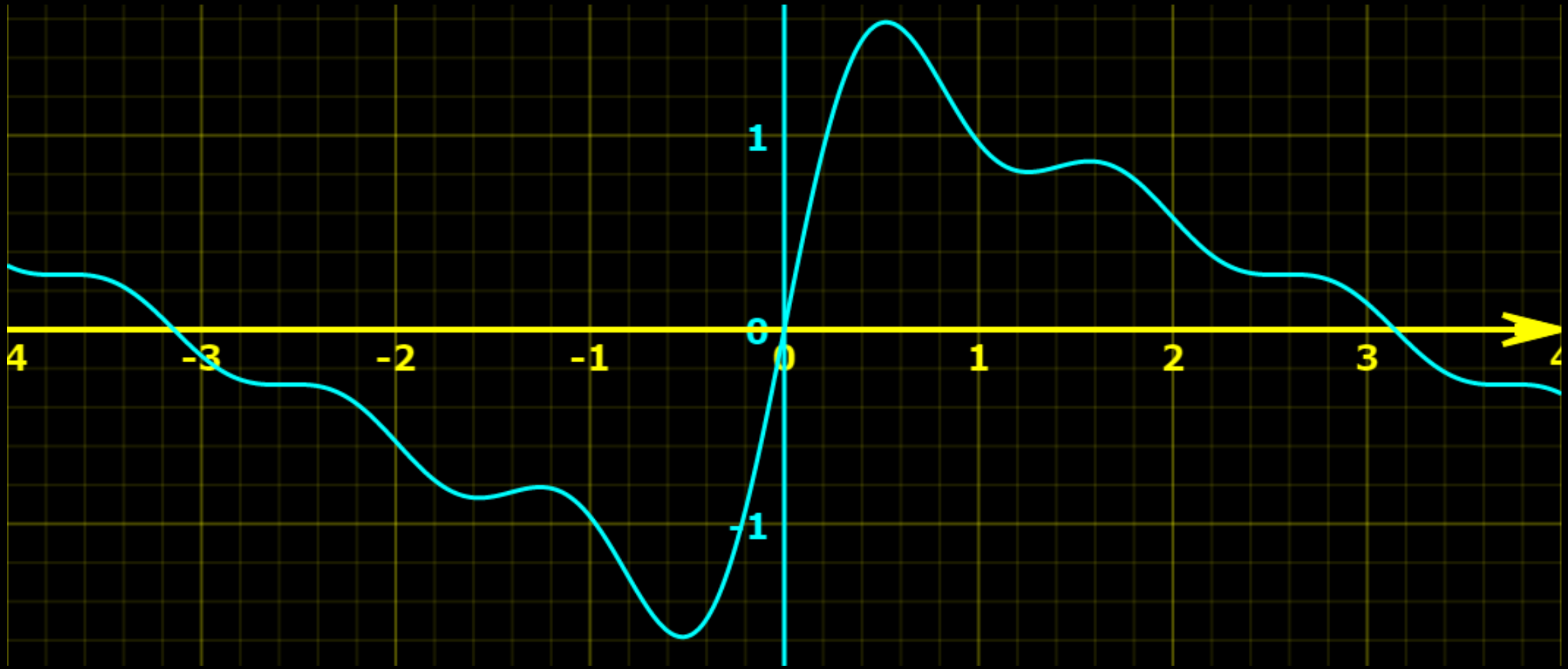

5 terms

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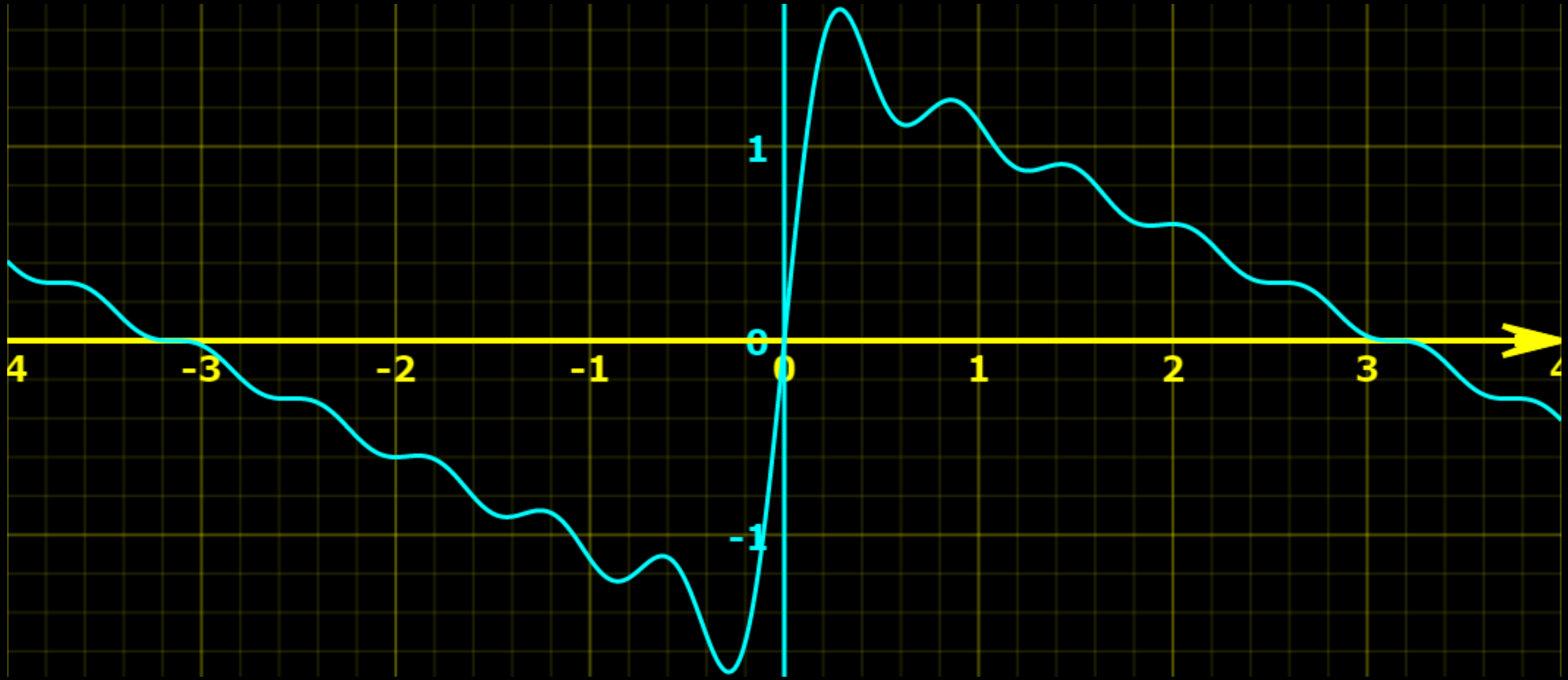

10 terms

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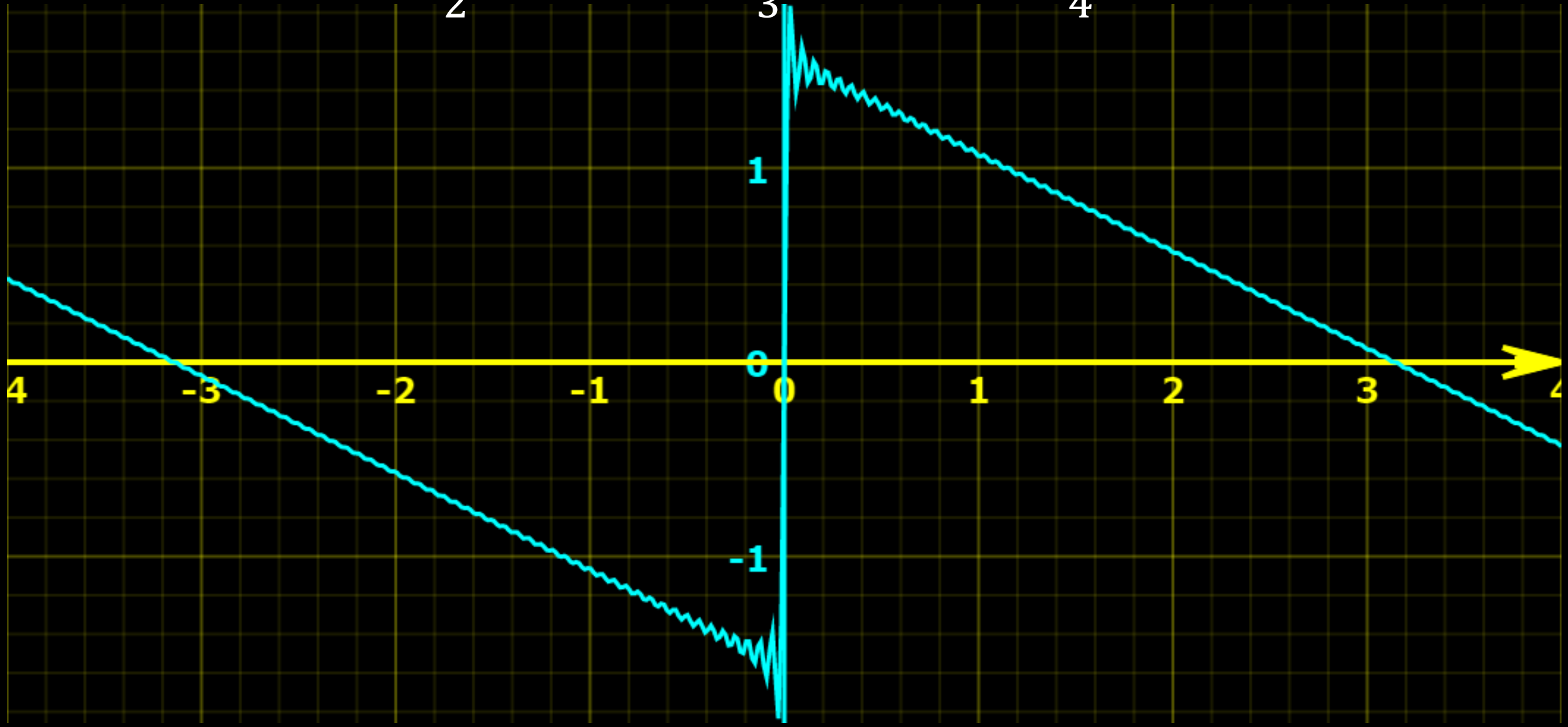
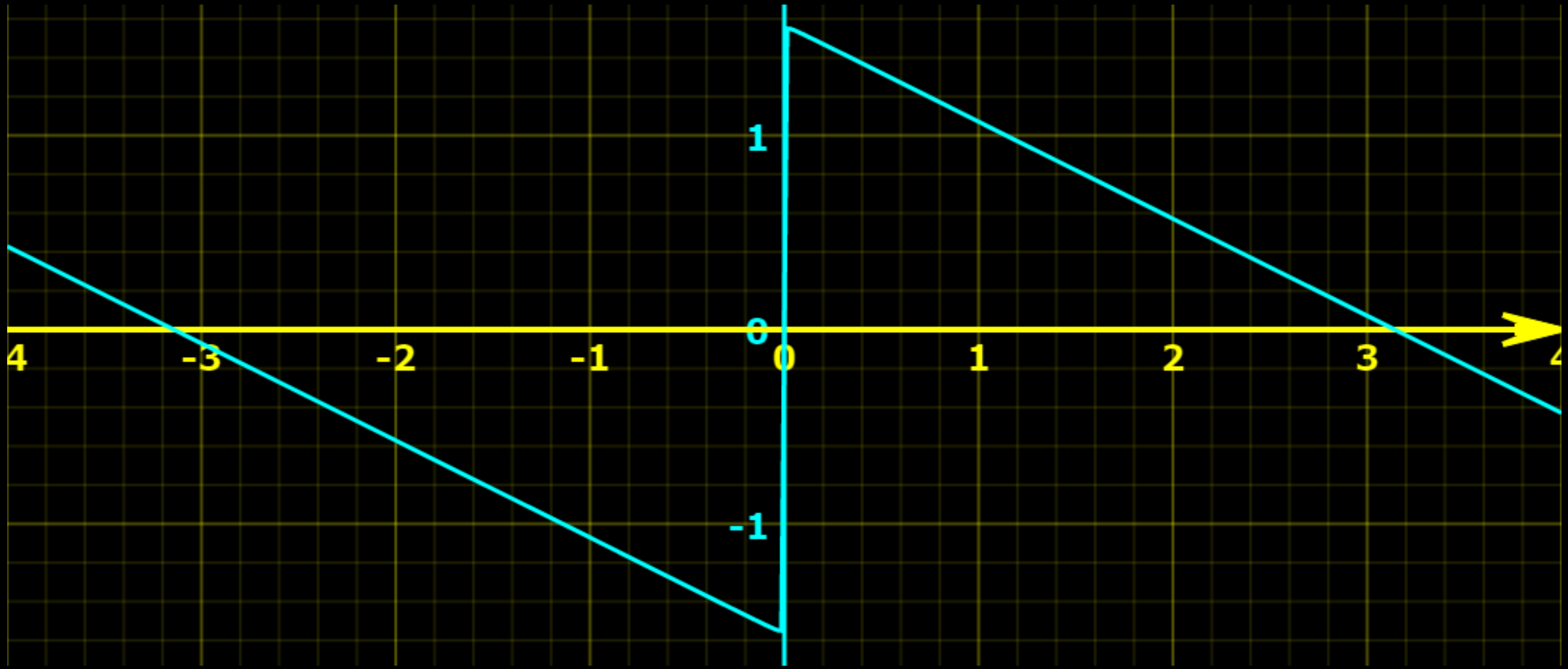


100 terms

What is $\sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x) + \cdots$?



5000 terms

Frequency and Harmony

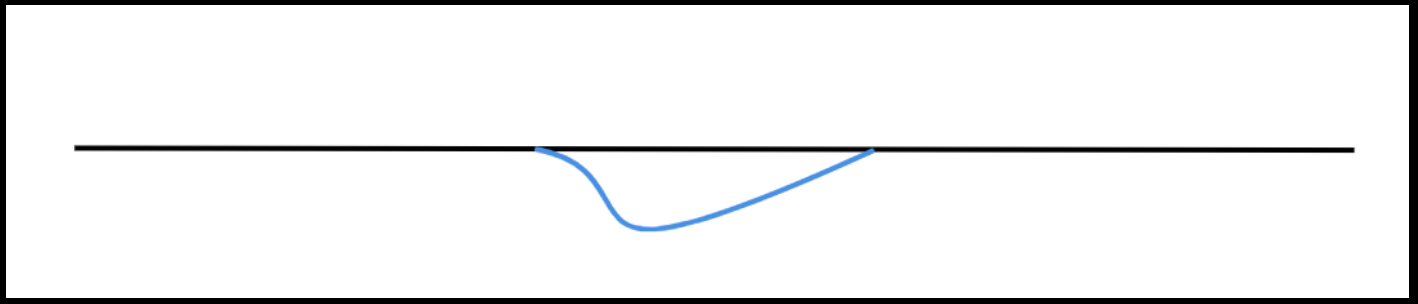
- Many worked on frequency and harmony (eg Galileo, Mersenne)
- Joseph Saveur (1653-1716) studied acoustics, detailed experiments on “the nodes of undulating strings”

The Wave Equation

- Take a string fixed at both ends (eg a violin string).
- Disturb it at time $t = 0$. The vertical displacement y at a point x along the string depends both on x and t.

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \times \frac{\partial^2 y}{\partial x^2}$$

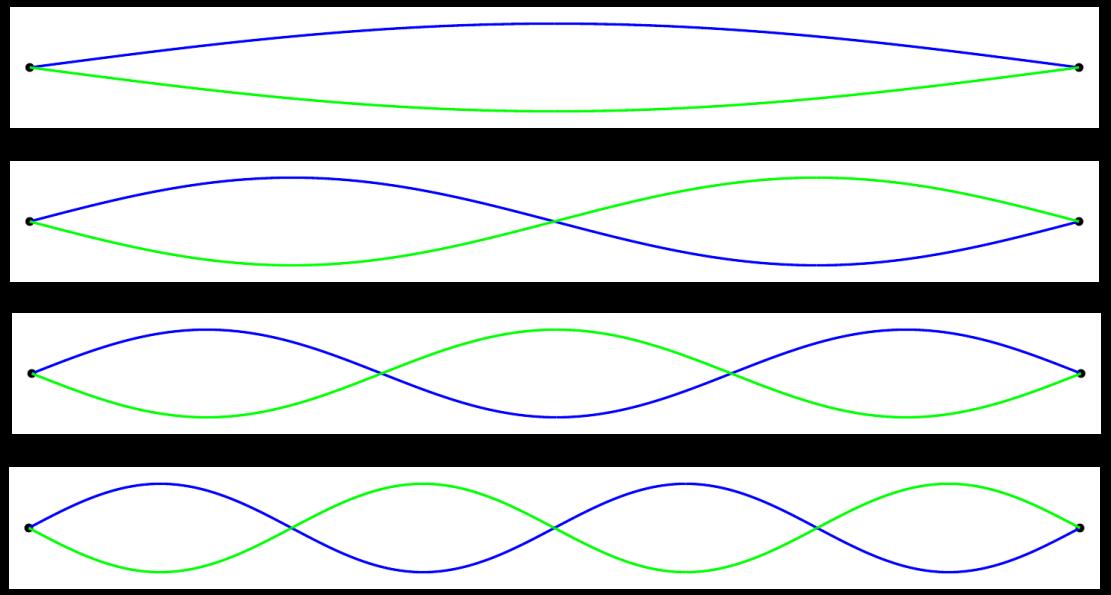
- Jean-le-Rond D'Alembert (1717-1783) found a method to solve this.

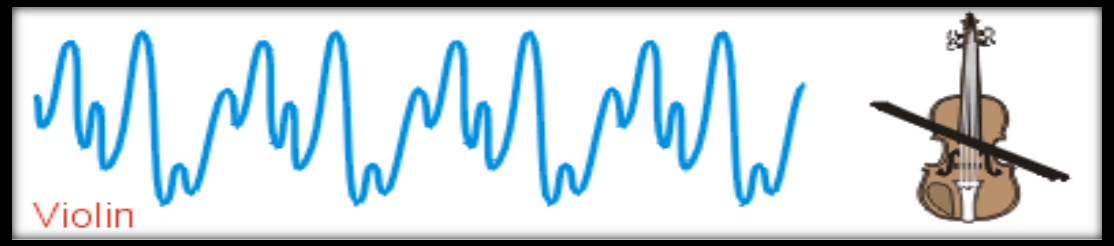


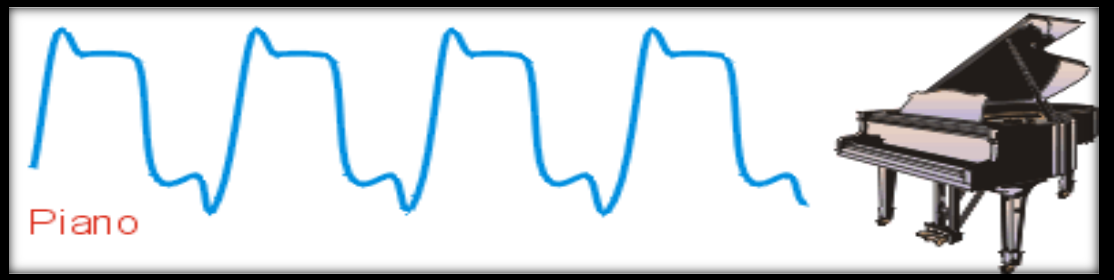
- Solution is wave A + wave B
- Periodic with period $2l$.

Wave $A \rightarrow$ $\leftarrow$ Wave B

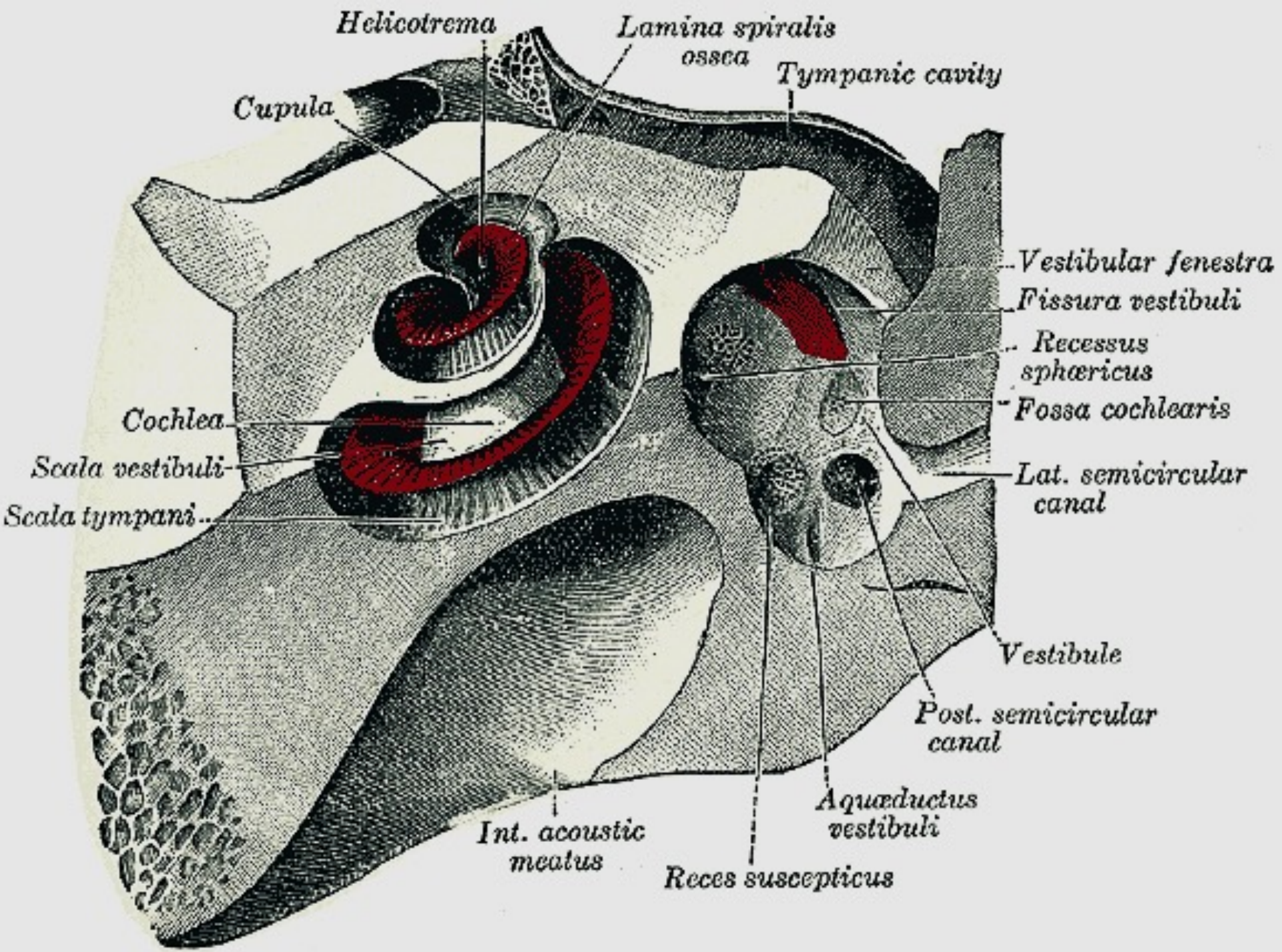
- Thanks to Fourier, we know every solution is a sum of sine waves of period $2l$ (l = length of string).
- Corresponds to frequencies $f, 2f, 3f$ etc.
- Instruments have different combinations of these waves.
- Initial “transient sound” is also important.







Violin/Piano soundwaves © Benjamin Hollis


Helicotrema
Lamina spiralis ossea
Tympanic cavity
Cupula
Vestibular fenestra
Fissura vestibuli
Recessus sphæricus
Fossa cochlearis
Cochlea
Scala vestibuli
Scala tympani
Lat. semicircular canal
Vestibule
Post. semicircular canal
Aquæductus vestibuli
Int. acoustic meatus
Reces susceptícus

GRESHAM COLLEGE

Gresham-LMS Lecture

The Maths of Gyroscopes and Boomerangs

Hugh Hunt

25th May, 6pm