The Maths of Coins and Currencies
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How can you detect a counterfeit coin? Why aren't coins square? Why is the wizarding money used in the Harry Potter universe so implausible? In this lecture we discuss the mathematics of coins. I'll focus on three main areas: creating, and testing for, the correct mix of metals, choosing the best coin shapes, and deciding on the ideal range of denominations.

Creating and detecting the real thing

Money has been used for thousands of years, and in that time there has been a huge range of materials in use, everything from gold and other precious metals to shells, leather, and of course paper. Many of these materials are "token" currency, in that they are not in themselves inherently of value. A five-pound note does not consist of five pounds worth of material. This is all very well if everyone agrees on the system. The advantage of gold and silver coins was that their value is intrinsic. If you have a gold coin then it is, literally, worth its weight in gold. Or at least, it is unless it is a counterfeit.

Counterfeiting is as old as trade, and one of the first stories about detecting it involves Archimedes. The story goes that the king (Hiero II of Syracuse, where Archimedes was from) had commissioned a golden crown and had given his goldsmith a certain amount of gold with which to make it. But Hiero suspected that the goldsmith may have in fact kept some of the gold for himself, making up the weight with silver. So, he asked Archimedes to find out the truth, without, of course, damaging the beautifully-wrought crown. An object made of a mixture of gold and silver, which is less dense than gold, would have a larger volume than one made of the same weight of gold. But it's surely impossible to measure the exact volume of so intricate an object as a crown. So how did Archimedes do it? Vitruvius, the Roman architect, wrote an account in his De Architectura (around 30 BCE) – it's a story we all remember: Archimedes sitting in his bath pondering the problem, suddenly realising that when he got into the bath he displaced exactly the same volume of water as the volume of his body, and then jumping out of the bath and running naked through the streets shouting "Eureka". The crucial realisation was that this is a way to measure volume without damaging an object (at least, not a metal one). Archimedes, goes the story, proceeded to take a block of pure gold the same weight as the crown, a block of pure silver the same weight as the crown, determine how much water they displace, and then compare how much water the crown displaced. With this information he could calculate exactly how much silver had been substituted. Legend has it that the goldsmith had indeed cheated the king.

There's an interesting postscript to this. Galileo, well over a thousand years later, wrote a short treatise on this problem, saying that he didn't think Archimedes would have used the method described by Vitruvius, as it's too imprecise. Galileo thought Vitruvius must have heard only that Archimedes had solved the problem using water, and guessed the wrong method. The issue is that the difference in the volumes with a small amount of silver mixed in is not very large, and there's a risk that experimental error might outweigh any true discrepancy. The density of pure gold is 19.32g/cm^3. Now, density $= \frac{\text{mass}}{\text{volume}}$, so volume $= \frac{\text{mass}}{\text{density}}$, which means that the volume of (for example) 1kg, or 1,000g, of gold is $\frac{1000}{19.32} = 51.76\ \text{cm}^3$. On the other hand, pure silver has a density of 10.49g/cm^3, meaning a kilogram of silver has a volume of 95.33cm^3. If the goldsmith had swapped 10% of the gold for silver, then a 1kg crown would consist of 900g of gold (volume 46.58cm^3) plus 100g of silver (9.53cm^3), for a total volume of 56.12cm^3, a discrepancy of

4.36cm^3. If you are measuring by marking off the water level rise in a vessel large enough to contain a crown, this 4.36cm^3 of water would be spread over the large surface area of the water in the vessel – it would be a negligible rise. But this isn't the only technology available. A better approach would be to have a vessel with a small hole in the side, filled up exactly to the level of the hole. Then when you put the crown in it, 56.12cm^3 of water flows out of the hole into a much narrower measuring tube that can be carefully calibrated. This method could certainly detect a difference of 4.36cm^3 cubic centimetres (water clocks of the time were accurate enough to do this, for instance). However, Galileo's idea for a method Archimedes might have used is rather ingenious, and illustrates a curious phenomenon. If you take equal weights of different metals (say gold and silver), then on a balance scale they will, well, balance. But if you now put that same pair of scales in water, they will no longer balance! Why? Well, let's imagine that the gold is on the left pan of the scale, and the silver is on the right. With both pans empty, underwater, there is the same amount of water pushing down on the left pan as on the right pan. So the scales balance. When we add the gold, there's the additional mass of the gold, but it has displaced its volume of water. So we subtract that amount of water. (If we have 1kg of gold, remembering that 1cm3 of water weighs 1g, the apparent weight of the gold would be $1000 - 51.76 = 948.24$g. On the right-hand pan, when we add the silver, we get the additional mass of the silver, which equals that of the gold, but we now subtract the mass of water in volume to the silver. The apparent weight of the silver would then be $1000 - 95.33 = 904.67$g. This means that the left-hand pan of the scales, with the gold in it, will have more mass pushing down on it, and so it will move downwards. This is a brilliant observation, and Galileo goes on to explain how to determine the exact mixtures of silver and gold by using submerged balances. Incidentally, even air weighs something – about 1.29g per cubic centimetre. To obtain truly equal weights we'd need to put our balance in a vacuum. If we measure in air, the denser substance will appear to weigh more. So perhaps the answer to that old trick question is that a pound of bricks really does weigh more than a pound of feathers!

The tale of Hiero's crown is an early instance of using the mathematics of density and volume to detect a forgery. The issue is somewhat simpler if you are comparing objects that are supposed to be the same size and composition, like coins. A suspect "gold" or "silver" coin can be weighed against one known to be genuine. It's also possible to determine the composition of a particular class of coin (say ones minted by a different country's mint) to determine exchange rates, by *assaying*. If you have a mixture of silver and gold plus other impurities you can determine the precise amounts by melting a sample, oxidising away the impurities, weighing what's left, which will be a mix of silver and gold, then dissolving the silver with nitric acid, to leave only the gold. You can then work out how much silver there had been, and what proportion of both of these were in the original alloy.

The other side of the coin, so to speak, is the challenge of the official mint: creating coins that have the precise legally required proportions of given metals. There's a nice paper by former Gresham geometry professor Norman Biggs on it if you are interested – the details are at the end of the transcript. An Italian mathematician and merchant called Leonardo of Pisa described techniques for producing alloys of a desired purity, or "fineness", in his extraordinary 13th century book *Liber Abaci*. You may not have heard of anyone called Leonardo of Pisa, but you probably have heard of him by his nickname Fibonacci. The so-called Fibonacci sequence was not invented by him, it had been known in India and elsewhere for a long time, but *Liber Abaci* was probably the first Western book to include it.

A typical coin-related problem Fibonacci describes is along the following lines: you have some rather impure silver bullion with fineness 4. (It doesn't really matter what units are being used, but just for the sake of argument let's say by "fineness 4" we mean that 4 ounces in a pound are silver.) and some very good quality bullion with fineness 9. And let's say the law requires silver coins to have fineness 7. How should the bullions be mixed in order to obtain the right fineness for the coin? The rule that Fibonacci gives is that since the "bad" bullion is 3 worse than you want, and the good bullion is 2 better, they should therefore be mixed in the ratio 2:3, that is, two parts bad bullion to three parts good. The reason this works is the following. If you take the fineness of the bad bullion to be A, the desired fineness to be B and the good bullion to be fineness C, then the recipe says to take $(C - B)$ of bad bullion for every $(B - A)$ of good bullion. Now, the total amount of alloy made here, for every C-B pounds of bad bullion used, will be $(C - B) + (B - A) = C - A$ pounds. The total amount of silver will be $(C - B)A + (B - A)C = CA - BA + BC - AC = B(C - A)$ ounces. Therefore, the fineness of silver in the alloy (in ounces per pound) is $\frac{B(C-A)}{C-A} = B$. This is a simple example of a much more general kind of problem, that's applicable beyond just coinage, and came to be known in English as *alligation*. It's relevant to many trades. You may be a winemaker who wants to blend wines of different unit costs to create one with a given cost. Or you may wish to know what possible combinations of a collection of differently priced items you can buy to reach a given total spend.

Fibonacci enjoyed puzzles, and in Liber Abaci he gives an alligation-related one called "*The three kinds of birds*". You are told that thirty birds are bought for a total of thirty pence (or denarii, in the original). There are three kinds of bird: partridges, pigeons, and sparrows. Partridge cost 3 pence, pigeons 2, and sparrows are two for a penny. What are the possibilities for how many birds of each kind are bought? Fibonacci tackles this ingeniously by translating it into a problem of alloys. The desired "alloy" is of fineness 1 (30 birds = 30 pennies, so on average 1 penny per bird). Partridges have fineness 3, pigeons 2, sparrows 0.5. If we pair partridge with sparrows, we see that partridges have 2 more fineness than desired, sparrows 0.5 less. So we should buy 0.5 partridges for every 2 sparrows. We can't buy half a partridge, so for every partridge we must buy 4 sparrows, a total of 5 birds. Similarly, for every pigeon we must buy 2 sparrows, a total of 3 birds. Pairing alloys like this and then combining pairs is one way to generalise allegation problems to more than two alloys. But most other authors in the centuries after Fibonacci rigidly said that the final alloy has equal amounts of each of the pairs. That wouldn't work for this puzzle, because doing this would force the total number of birds to be a multiple of 8. Fibonacci's flexibility not only allows him to think of this puzzle apparently totally unrelated to alligation, but also to find a solution. Remember the total number of birds bought is 30, so the problem reduces to the ways of making 30 as a sum of whole numbers of 5's and 3's. Assuming we have at least one of each bird, the only possibility is three 5's and five 3's. That is, 3 partridges (which imply 12 sparrows) and 5 pigeons (which imply 10 sparrows). In sum: 3 partridges (total 9p), 5 pigeons (total 10p), 22 sparrows (total 11p), giving 30 birds for 30 pence. Clever! If we didn't have the restriction of whole numbers of birds, there would actually be infinitely many solutions to this puzzle. Today, the study of problems like this has developed into a subject known as linear programming.

The shape of money

Now we know how to make a coin of the desired composition. But what does it look like? Most coins through history have been circular, though not all. There are examples of triangular and square coins, and other polygons with up to sixteen sides, as well as coins with round, square, and hexagonal holes. There are also rectangular and oval coins. Holes are useful for storing coins on strings or necklaces. Originally, coins would have been basically rounded lumps of metal. Then, when they began to be stamped with for example the Emperor's profile, they became flat. The round shape naturally forms a roughly circular shape when impressed with a stamp like this. A circular shape is more robust than one with lots of sharp corners, as these would wear down more quickly or possibly break off. A final reason is a more modern one. Vending machines find it easier to differentiate between different coins if each coin has a fixed diameter that's different from all the other coins. Circles, of course, almost by definition, are shapes with a constant width – their diameter at any point is exactly twice the radius. (This is useful for coins because of vending machines, but it also explains why most manhole covers are round. You don't want the cover to accidentally fall down the hole. With a circular cross-section, there is no way it can be aligned to be any narrower, so you are safe from that calamity.) But the circle is not the only shape with this property.

Take an equilateral triangle, and suppose each side length is s. Replace each of its edges with an arc of a circle whose centre is the opposite vertex, and whose radius is s. Imagine rolling this shape along a line. To begin with, vertex A is directly above vertex B, and the height of the shape is s. Each curved side is a circle arc, so as we roll along from B to C, the highest point of the shape remains A, and the height remains s. When we reach C, the highest point is still A, which is now vertically above C, and now C becomes the pivot, and the top part of the shape rolls along the circle arc from A to B, still maintaining that height s. So, in fact, this shape would fit exactly in a passage of constant height s.

The same reasoning works for any higher odd number of sides. We can have a "pentagon of constant width", and also (pertinently for money) a "heptagon of constant width". This is exactly the shape of British 20p and 50p coins. (I have a soft spot for the 20p coin, which was introduced in 1982, because when I was a child it was the coin the tooth fairy brought me. My own children got one-pound coins: there's inflation for you.)

These curved shapes I've just described are called Reuleaux polygons (after the German engineer Franz Reuleaux who investigated their possible mechanical applications). The technique only works for odd numbers, because regular polygons with an even number of sides have the problem that every vertex is opposite another vertex, and the distance between two opposite vertices is always going to be greater than any other cross-sectional slice, however much you bulge out the sides. Reuleaux polygons do have some applications outside of coinage, too. For example, one property they have is that they can make a complete rotation inside a square such that at all times every edge of the square is in contact with the shape. A

Reuleaux triangle drill bit will produce a very nearly square hole. This isn't just theoretical. One such drill bit was patented by Harry Watts in 1916 – it needs a special chuck to allow the centre of rotation of the bit to move, but with that in place, it covers over 98% of the square.

As well as the heptagonal coins in UK currency, there's also the Canadian one-dollar coin (nicknamed the "loonie" because it has a picture of a loon on one side): this is an eleven-sided Reuleaux polygon. Some Bermudian coins are Reuleaux Triangles. There's the 1997 one dollar coin, as well as several "Bermuda Triangle" commemorative coins.

This is cute, but why do it? Why not just make circular coins – it's easier after all? Of course, it's just fun to have a bit of variety. It's also harder to counterfeit a Reuleaux polygon than a simple circle, but there's one possible other reason, and it's to do with areas.

Firstly, there's a theorem called Barbier's Theorem that says all shapes of constant width s have the same perimeter, namely πs. This is fairly easy to see for circles (diameter s means circumference πs) and Reuleaux polygons, whose perimeters are arcs making up half the circumference of a circle of radius s, but far from clear that it holds in general. However, it does. On the other hand, the areas of these shapes can vary. A Reuleaux triangle with constant width s can be thought of as the superimposition of three 1/6$^{\text{th}}$ slices of a circle of radius s. So you can find the area by adding three of these together, but then remembering that this counts the equilateral triangle inside three times. So the formula for the area is

$$3\left(\frac{1}{6}\text{circle}\right) - 2(\text{triangle}) = \frac{1}{2}(\pi s^2) - 2\left(\frac{1}{2} \times s \times \sqrt{s^2 - \frac{1}{2}s^2}\right) = \frac{1}{2}\pi s^2 - \frac{\sqrt{3}}{2}s^2 = \frac{1}{2}s^2\left(\pi - \sqrt{3}\right) \approx 0.7s^2$$

This is less than any other Reuleaux polygon and also less than a circle of diameter s, which would have area $\frac{1}{4}\pi s^2 \approx 0.79s^2$. In fact it can be shown that the Reuleaux triangle has the smallest area of any shape of constant width s. So, in terms of material used, for a given width you save money with the Reuleaux triangle. You might like to work out the area of a Reuleaux pentagon with constant width s. Is it better or worse than a circle?

Of course, anyone familiar with British coins will have spotted that not all coins can possibly have constant width, because we have a twelve-sided coin: the £1 coin. This has an even number of sides, so it can't (and indeed doesn't) have constant width. So, what's going on? It's not the first twelve-sided coin in British currency. Many years ago the threepenny bit had twelve sides. However, you'll notice that the modern £1 coin does have slightly curved sides. This makes the discrepancy between its smallest (23.03mm) and largest (23.43mm) widths small enough for vending machines to cope. The old threepenny coin would have caused more issues (if you wish, you can calculate the difference between smallest and largest width to confirm this), and that is why the curved sides were adopted.

So, the coin is a "near enough" design that satisfies vending machines. That doesn't mean I have to like it.

Coin values and denominations

Now we have coins made of the right alloys and the right shape. Our final currency decision is what denominations they should have.

Essentially everyone has decimal currency nowadays – in other words, a currency where units are subdivided into powers of ten sub-units. For example, Euros, and US dollars, are divided into 100 cents; pounds sterling are divided into 100 pence. The Nigerian Naira is 100 Kobo, one Russian rouble is 100 kopeks (and has been for over 300 years). And so on.

There are only a handful of exceptions, mostly where the subdivisions are now too small to be used, or where the currency is not in use for other reasons. For example the *scudo* is still the official currency of the Sovereign Military Order of Malta, though everyone uses Euros and has done for many years. The scudo is divided into 12 *tari*, with 1 *tari* equivalent to 20 *grani*.

But historically, the situation was reversed – almost every currency was non-decimal. Why? And why did it change? Just as an example, if we look at historical British money, we had shillings divided into twelve old pennies, and twenty shillings in the pound, or sovereign. (And 21 shillings in a guinea.) Pennies in turn were divided into four farthings (from the word "fourthling"). They were legal tender until 1960. So, you can divide up a shilling into two, three, four, six whole numbers of pennies, and into greater subdivisions with halfpennies and farthings. A few hundred years ago there were coins worth ¼ penny (the farthing), ½ penny, 1 penny, 2 pence (the half groat), threepence, fourpence (groat), sixpence, a shilling, half a crown

(2 shillings and sixpence), a crown (five shillings), and gold sovereigns (1 pound). This is simplifying matters slightly because the values of the silver and gold coins fluctuated with the prices of these metals – that's why guineas ended up being worth slightly more than a pound – and the purity of the coins; silver coins had ever-increasing amounts of copper, with the currency being repeatedly debased because of it. The values weren't formally fixed until later.

You could divide a pound equally, using whole numbers of pennies, into 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80, 120 and 240. There's not as much flexibility in the decimal pound, which can be divided equally into 1, 2, 4, 5, 10, 20, 25, 50, and 100. That's the big advantage of pre-decimal shillings and pounds. The big disadvantage is the arithmetic. Once you start writing down calculations, it is very cumbersome to use our decimal number system for calculations in a system that has twelve pence to the shilling and twenty shillings to the pound. (Consider also our baffling imperial units of sixteen ounces to a pound and fourteen pounds to a stone, but obviously (!) twenty fluid ounces to a pint in the UK, and sixteen in the US. Our American friends have a rhyme for converting water volume to weights: "a pint's a pound the world around". Only it isn't because in England "a pint of water weighs a pound and a quarter".)

If you think pre-decimal currency is bad, spare a thought for Harry Potter and his Hogwarts friends. In J.K. Rowling's books, wizard money consists of golden galleons, silver sickles and bronze knuts. There are 17 sickles in a galleon, and 29 knuts in a sickle. This is an insane system that wouldn't even allow amounts like half a galleon, never mind the formidable arithmetic that would be required to make transactions.

Back in the real world, people have been making the case for decimalisation for hundreds of years. In 1696, Christopher Wren, former Gresham Professor of Astronomy, proposed the introduction of a silver "noble", divided into 10 "primes" and 100 "seconds", "which Centessimal Division will be very proper for Accounts". The objections to this idea and others were mostly that, desirable though it would be, people would never be able to learn a new system. Of course, if a country is brand new that's less of a problem. The United States dollar, introduced in 1792, was divided not into "pieces of eight" like the Spanish silver dollar from which it took its name and initial value, but from the start into tenths (dimes) and hundredths (cents). The debate in Britain rumbled on, and in the 19th century several commissions and a lot of parliamentary time was taken up with it. The florin (two shillings) was introduced in 1849 as a possible precursor to decimalisation, because it's a tenth of a pound (and some of them – the so-called Gothic Florins – even had "one tenth of a pound" written on them). There was even a double florin, briefly, which is equivalent to our modern 20 pence piece, because it is a fifth of a pound. The double florin, though, worth four shillings, was very close in size to, and easily confused with, the five-shilling crown, so it was withdrawn. It was only minted for four years, from 1887-1890 [15]. After yet more Royal Commissions, including in 1918 and 1961, decimalisation was finally agreed, and the big switchover happened on 15th February 1971.

So, what coins should we have? It makes sense for whole numbers of smaller coins to go into the pound. This means the possibilities for coins are 1, 2, 4, 5, 10, 20, 25, 50. In the UK we have six of these eight: 1p, 2p, 5p, 10p, 20p, 50p (as well as £1 and £2 coins). A 4p coin would not be very useful – it can already be made with two 2p coins, and is too close in value to the 5p. In the US, they have fewer coins: 1 cent, 5 cents (nickel, though at 75% copper and 25% nickel, perhaps it should be called a copper), 10 cents (dime), and 25 cents (quarter). There is a dollar coin but, due to the enduring popularity of the dollar bill, it hasn't caught on.

Is one system better than the other? The best systems combine simplicity (a small number of different denominations) while allowing change to be given using a small number of coins. The simplest system – just have 1p coins – would require us all to lug great big bags of coins around all the time. At the other extreme, if there were a coin of every value we would need gigantic cash registers for all the different coins and would no doubt never have a 47p coin when we needed one. As a measure of how efficient different systems are, Dr Adam Townsend at the University of Durham suggested using the average number of coins required to make up each amount of change up to the point where the currency switches to notes. For the US, that's one dollar. For the UK, it's five pounds. We can think about whether this is the right measure or not. Some currencies have notes starting at much lower values than others, and this may skew the results. I modified the Python script he recommended to calculate not just the mean but the median number of coins, and I also felt that a US – UK comparison would be more informative if we calculated the change up to the same amount in both currencies: 99p/99¢. My code, and links to his articles, are at the end of this transcript.

For values of change from 1p up to and including 99p, the current UK system needs a (mean) average of

3.43 coins, and a median of 3 coins. For up to 99¢ the US system needs a mean of 4.75 coins and a median of 5. So it's much worse, with some amounts needing eight or even (for 94¢ and 99¢) nine coins (the UK system never needs more than 6 coins). Adam Townsend calculates average change up to and including the smallest bill so it calculates 4.75 for the US system but 4.61 for the UK.

By the way, if we had eight fingers like most cartoon characters, rather than ten, we'd probably use a base 8 system with 8×8 pennies in the pound. Then we could have 1, 2, 4, 8, 16 and 32p coins, and with a £1 =64p and £2 = 128p coin, we'd need only a mean (and median) of 4 coins to make change up to the smallest note being a £4 = 256p note, and you'd need at most one of each coin for any given amount of change. But I digress!

Should the US switch to 20 cents rather than quarters? Then the coins are 1, 5, 10, 20. The mean average change is 5.05 coins, so the US should stick with quarters, unless it wants to add other coins, like a 2 cent or 50 cent coin. Does this mean the should UK switch to "quarters" rather than 20p? Curiously, this would make absolutely no difference to the amount of change received. The outcome is identical, given the other coins we have.

All this assumes we are getting change in the most efficient way. Anyone who has used an automated checkout at a supermarket knows that's not always the case. Because self-checkout machines are often based on global designs, and most countries have fewer different coins than us, these machines can take any of our eight coins but can usually only give six different kinds of coins in change. Most often, this is 1p, 2p, 5p, 20p, £1, £2. If we just look at amounts of change up to 99p, we now find we get 4.75 coins on average (or a median of 5), which is much more than we would get from a human. If we test amounts up to £4.99, we get a mean of 5.91 and a median of 6. Interestingly, as Adam Townsend points out, things would be much better if the coins instead were 1p, 2p, 5p, 20p, 50p, £2. For amounts up to 99p you would get a mean of 3.84 coins and a median of 4.

There are some issues with coins. The 5p is irritatingly small. The 1p and 2p are irritatingly heavy. (If you are outside the UK, you probably have similar gripes with your own currency.) Is it time to ditch some of them? We used to have halfpenny coins, even post-decimalisation, and we even had farthings within living memory. The farthing was discontinued in 1960 and the halfpenny in 1984. Now, I've done some calculations, with the aid of the Bank of England's inflation calculator (at time of writing, as it only covers complete years, the calculator only compares a given year to 2021 prices.) It turns out that £1 in 1960 would be worth £16.18 today (or actually at the end of 2021). One farthing is a quarter of an old penny, and so 1/960th of a pound. So that farthing would be worth 1.7 new pence as at 2021, and with inflation at over 10% in 2022, that's only going up. By that token we should get rid of the 1p coin and we are close to the point where we might consider getting rid of coppers completely.

How about halfpenny coins? They were discontinued in 1984, and £100 in 1984 is worth £263.02 today. So a 1984 halfpenny would be worth 1.3p today. Again, a strong case for getting rid of the 1p coin. (In fact, halfpennies would have been phased out sooner, except that the Bank of England were worried that it might worsen inflation, as all the prices might be rounded up to the nearest penny.) The halfpenny coin was not missed as a coin in itself, but some had other uses for it. A letter to the Times begged "let not its existence be imperilled. It is indispensable for levelling off pendulum clocks". We would manage perfectly well without the penny. The Canadian mint ceased production of the Canadian cent (fondly known as the penny) in 2013. They didn't have a two cent coin so their smallest coin is now 5 cents. Items can still be priced in pennies, and electronic transactions can still be done to the nearest penny, but change is given to the nearest five cents.

One day we may not use coins at all. But I think that day is still (happily) quite far off. I hope you've enjoyed this mathematical guide to coins. In my next lecture on "maths and money" we look at how maths can help us to find the best strategies when we buy, sell, bargain, barter, and bid at auctions. I hope to see you then.

Further Reading

- Hiero's Crown: for a description of how the account of Vitruvius could have been correct, see *The Vitruvius Tale of Archimedes and the Golden Crown*, by Amelia Carolina Sparavigna, https://arxiv.org/ftp/arxiv/papers/1108/1108.2204.pdf. Professor Chris Rorres at Drexel University has posted the relevant passages of Vitruvius, the treatise of Galileo, and other sources, alongside English translations, at https://www.math.nyu.edu/~crorres/Archimedes/Crown/CrownIntro.html
- For a much more in-depth history than I can give on alligation and linear programming, read the excellent paper *Linear programming from Fibonacci to Farkas*, by Norman Biggs, which you can download at http://eprints.lse.ac.uk/106596/.
- If you want to read Liber Abaci and your Latin is on the rusty side, there's an English translation (we only had to wait 800 years!) *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, by Laurence Sigler (Springer New York; 2002, ISBN 978-0387407371).
- Marianne Freiberger wrote a good article in 2017 about the introduction of the 12-sided £1 coin in Plus magazine: https://plus.maths.org/content/new-1-coin-gets-even
- You can read a short summary of the history of decimalization at the Royal Mint website: https://www.royalmint.com/discover/decimalisation/decimal-debate/
- Adam Townsend's entertaining blogs about coin denominations in different currencies and about vending machine change appeared in Chalkdust magazine.
  https://chalkdustmagazine.com/blog/forget-a-new-1-pound-coin-we-need-a-1-pound-23-coin/
  https://chalkdustmagazine.com/blog/self-service-machines-give-awful-change/
  He also made a programme on change for Radio 4's "Boring talks" about it, which is well worth a listen: *Boring Talks Episode 33: Change* https://www.bbc.co.uk/programmes/p06xvmpz
- The Python script I used is adapted from the one Adam Townsend used. My script is below, and you can play around with it at one of the many free online Python compilers, for example https://www.w3schools.com/python/trypython.asp?filename=demo_compiler

```
coins = [1,2,5,20, 100, 200]
rangecoin=99
min_coin = [100] * (rangecoin+1)
min_coin[0] = 0
for min_of_i in range(rangecoin+1):
 for c in coins:
  if c <= min_of_i and (min_coin[min_of_i - c] + 1 < min_coin[min_of_i]):
              min_coin[min_of_i] = min_coin[min_of_i - c] + 1
print(min_coin)
print(sum(min_coin)/rangecoin)
import statistics
print(statistics.median(min_coin))
```

- The Bank of England's inflation calculator https://www.bankofengland.co.uk/monetary-policy/inflation/inflation-calculator can show how prices compare to today, for any year since 1209!

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