Can Maths save the whales and cure cancer?

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On the planet Earth, man had always assumed that he was more intelligent than dolphins because he had achieved so much - the wheel, New York, wars and so on - whilst all the dolphins had ever done was muck about in the water having a good time. But conversely, the dolphins had always believed that they were far more intelligent than man - for precisely the same reasons.

Douglas Adams, The Hitchhiker’s Guide to the Galaxy

Top challenges for human kind in the 21st Century:

Curing cancer

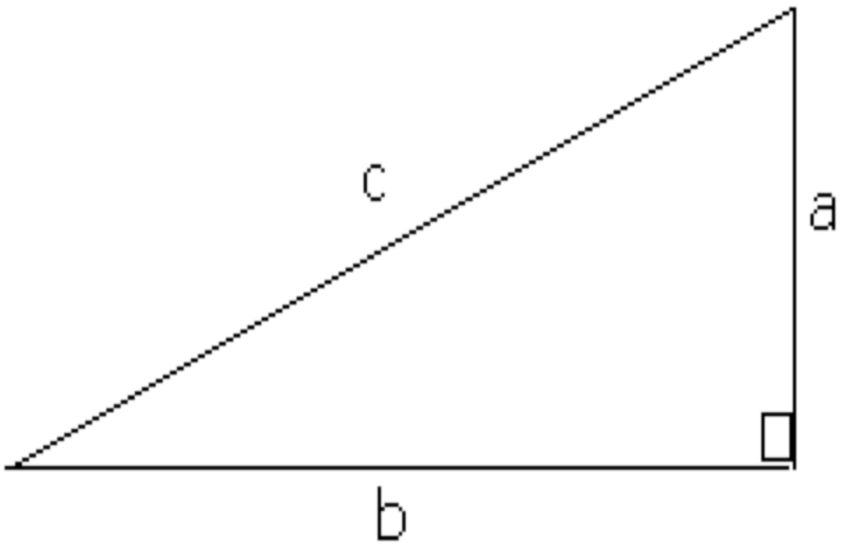
Saving the environment

Only two of the problems for which any possible solution involves the applications of mathematics

Mathematics on its own will not solve any practical problem!

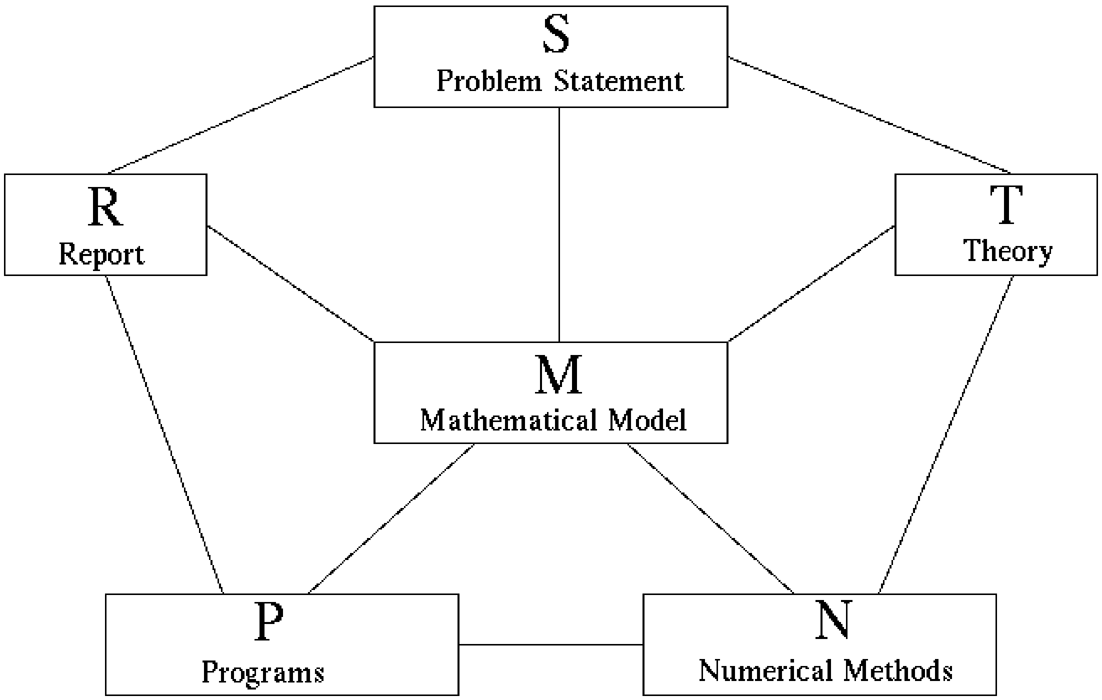
The main requirement of mathematics is that it should be:

- self consistent with its axioms, and
- it should lead, by rigorous arguments, to amazing theorems eg. Pythagoras' Theorem which represent eternal truths

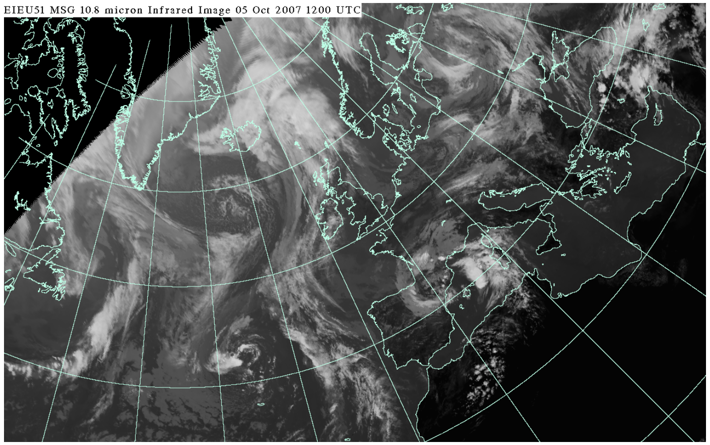


To apply mathematics to the real world, mathematicians must work with scientists/engineers, to turn real life problems into mathematics, and to then use (often very sophisticated) mathematics, to solve the resulting equations

We call this process mathematical modelling

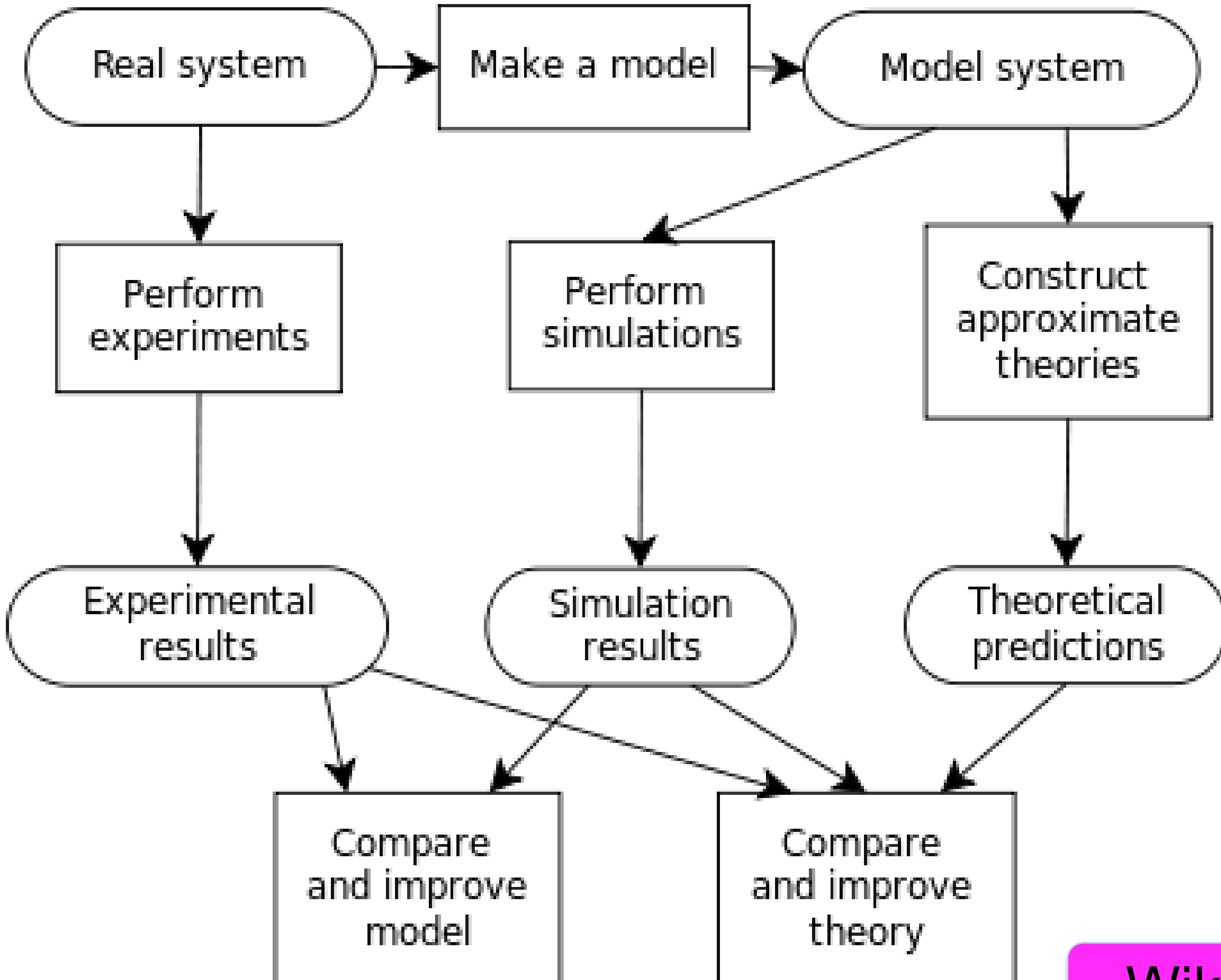


Example of mathematical modelling: the weather forecast



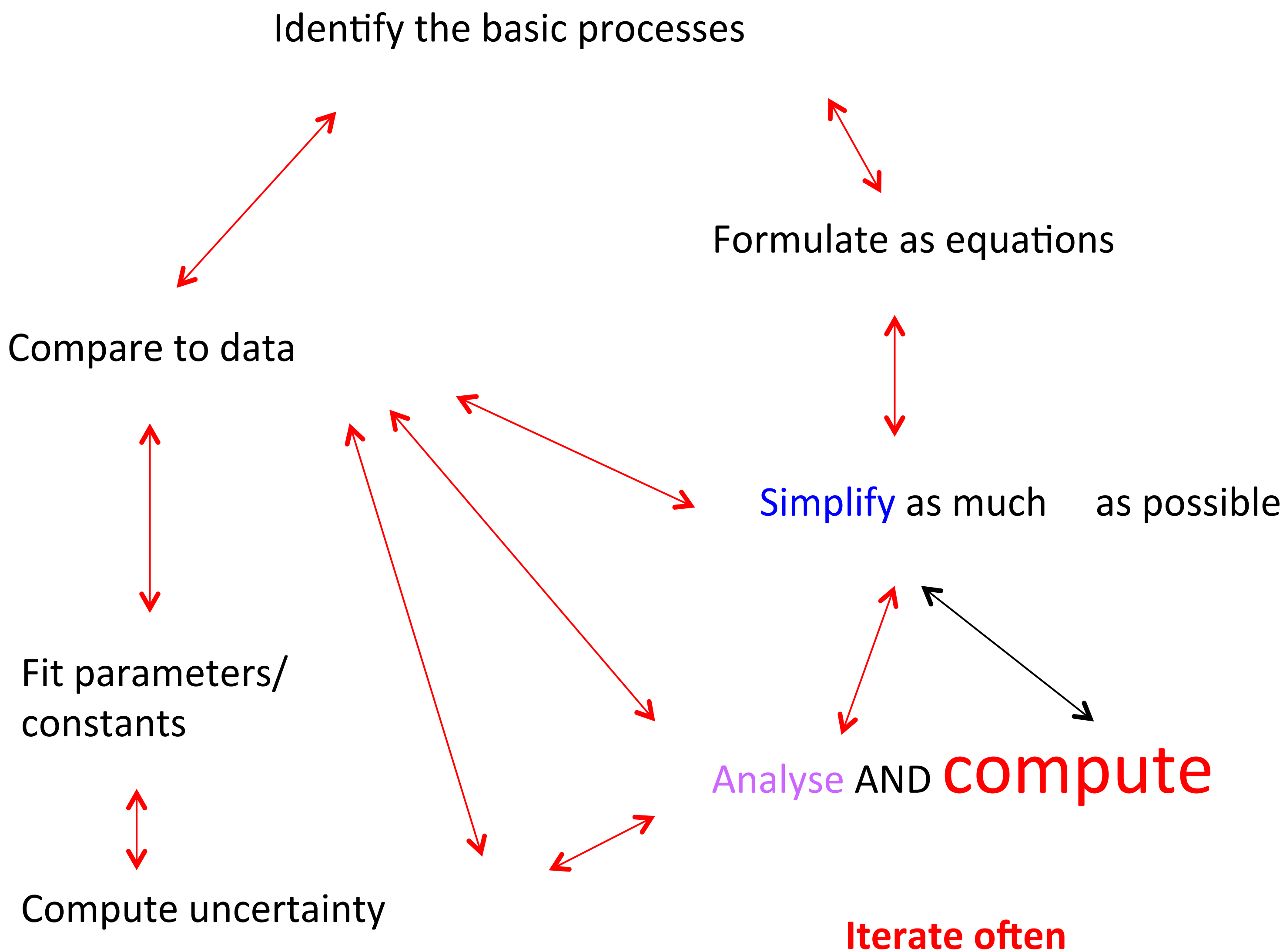
- Start with the laws of physics
- Encode as (differential) equations
- Use data to determine today’s weather accurately
- Use a super computer to solve the equations for 24 hours to give us the weather tomorrow
- Continuously update the forecast by assimilating new data
- Present the results in a way that all can understand.

The 'official' Modelling/Simulation Process

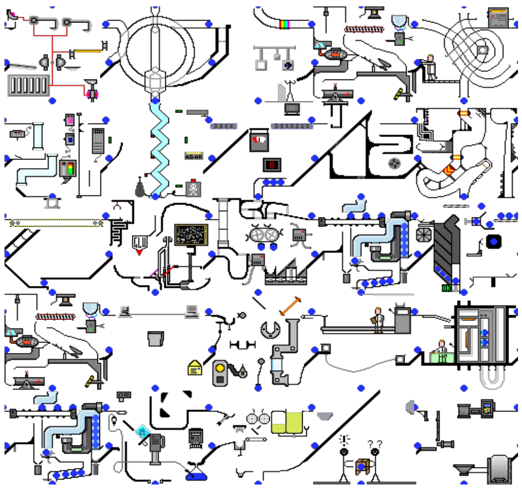


Wikipedia

My own experience!



Simulation



Try to get all of the details as right as possible so that the conclusions are as accurate as possible

Vast computer codes with billions of lines

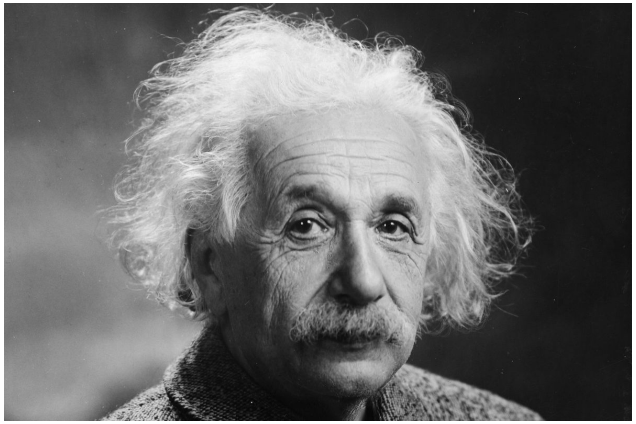
Slow to run and hard to change

Major investment of time and resources

Built up over many years of experience

Model

A simplification to a small system of equations, which capture its essential essence and are simple enough to make analytical calculations

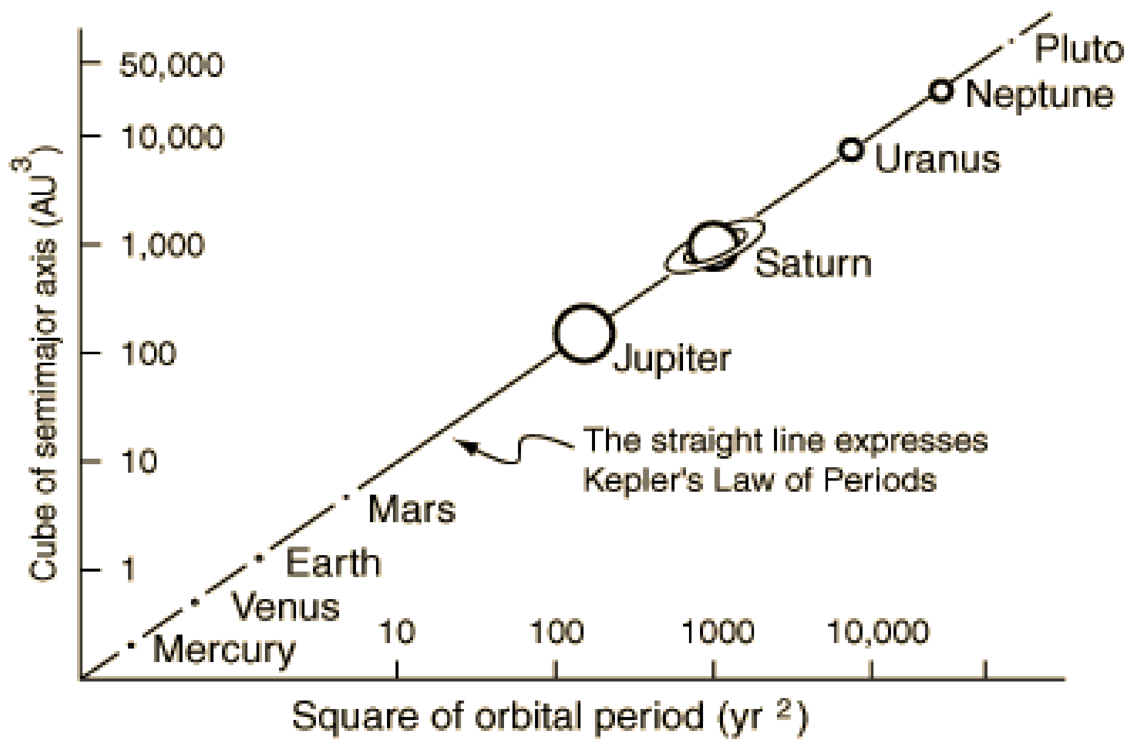


A model should be as simple as possible, And no simpler!

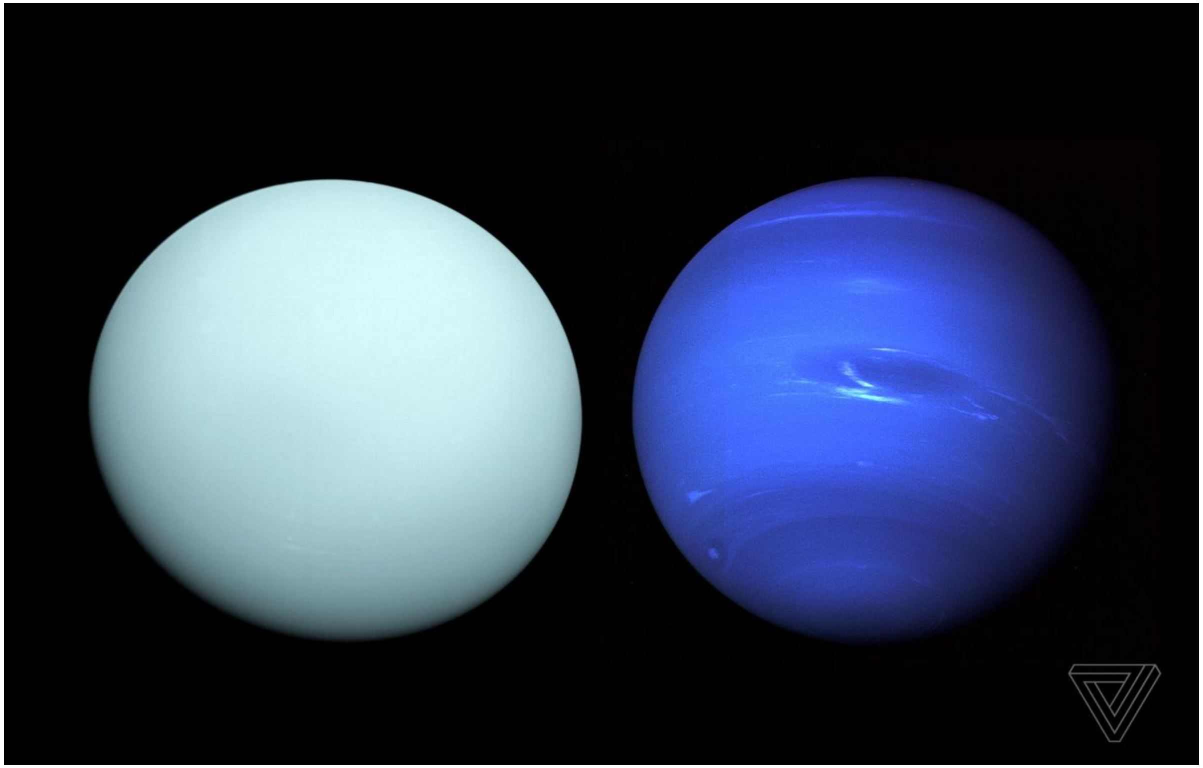
A. Einstein

A formula derived from an analytical calculation can give a clear view of the role of the parameters in that system without having to run a very large number of calculations

Eg. Newton's model of the solar system

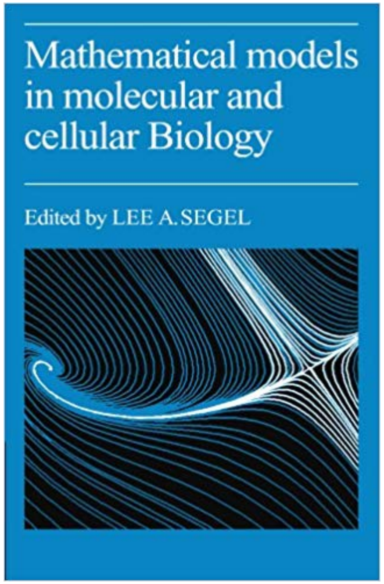


A great modelling success: The Discovery of Neptune



Model predicted something new and unexpected

Mathematical models are now widely used:



Physics and engineering eg. Newton's laws

Chemistry eg. Molecular dynamics

Biology eg. Mathematical biology, desease

Economics eg. Game theory

Sociology eg. Crowds ??? Psychohistory

Also by irresponsible newspaper journalists

$$x = \frac{f * l + n * o}{p}$$

A better example: A simple climate model

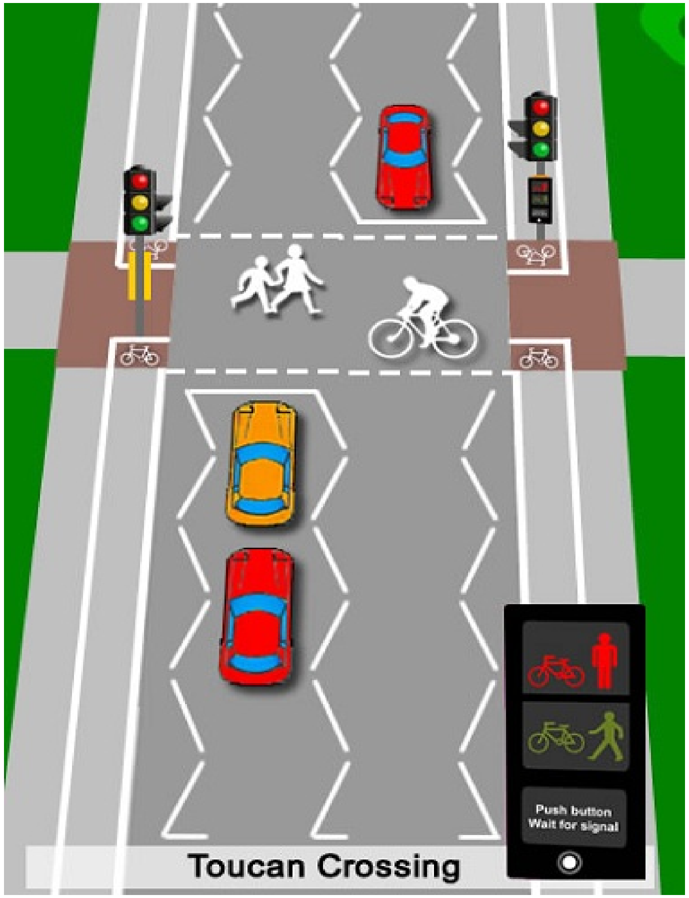
T : average Earth temperature
S(t): solar radiation
a : Earth's albedo
e : Average atmospheric emissivity

Balance energy from the sun with energy lost from the Earth to give:

$$T = \left(\frac{(1-a)\ S(t)}{\sigma e} \right)^{1/4}$$

Allows simple predictions of the change of temperature with Carbon Dioxide and ice melt

Worked Example: How did the mathematician cross the road?



What is the ‘optimal’ delay on a pedestrian crossing light?

A simple set of initial assumptions:

1. Pedestrians arrive at a steady rate at the crossing

2. Traffic arrives at a steady rate

3. It takes the pedestrians a time T to cross the road

4. The optimal delay time is the one that maximises the average total flow F of both the pedestrians and the traffic.

In any time interval of length t

a*t pedestrians arrive

b*t cars arrive

Lights are green

All the waiting pedestrians cross

Lights are red

Wait a time 1/a before a pedestrian arrives and presses the button to cross. The lights then have a delay D before they go green again.

We have a complete cycle of time 1/a + D+T

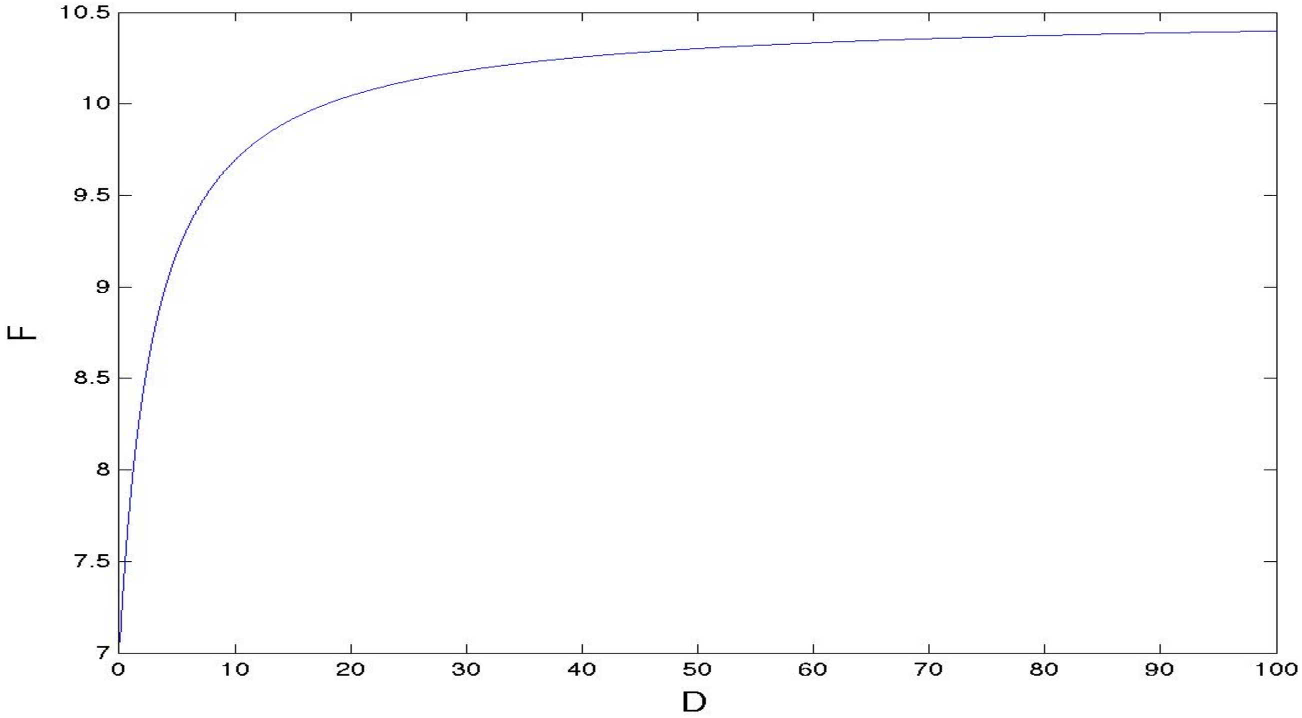
1 + a*D pedestrians arrive and then cross

b*(1/a +D) cars cross

Average flux of people and cars:

$$F = \frac{1 + a\,D + b\left(\frac{1}{a} + D\right)}{\frac{1}{a} + T + D}$$

$a = \frac{1}{2}$, $b = 10$, $T = 1$



Solution:

Make D as large as possible, until the patience of the pedestrians gives out

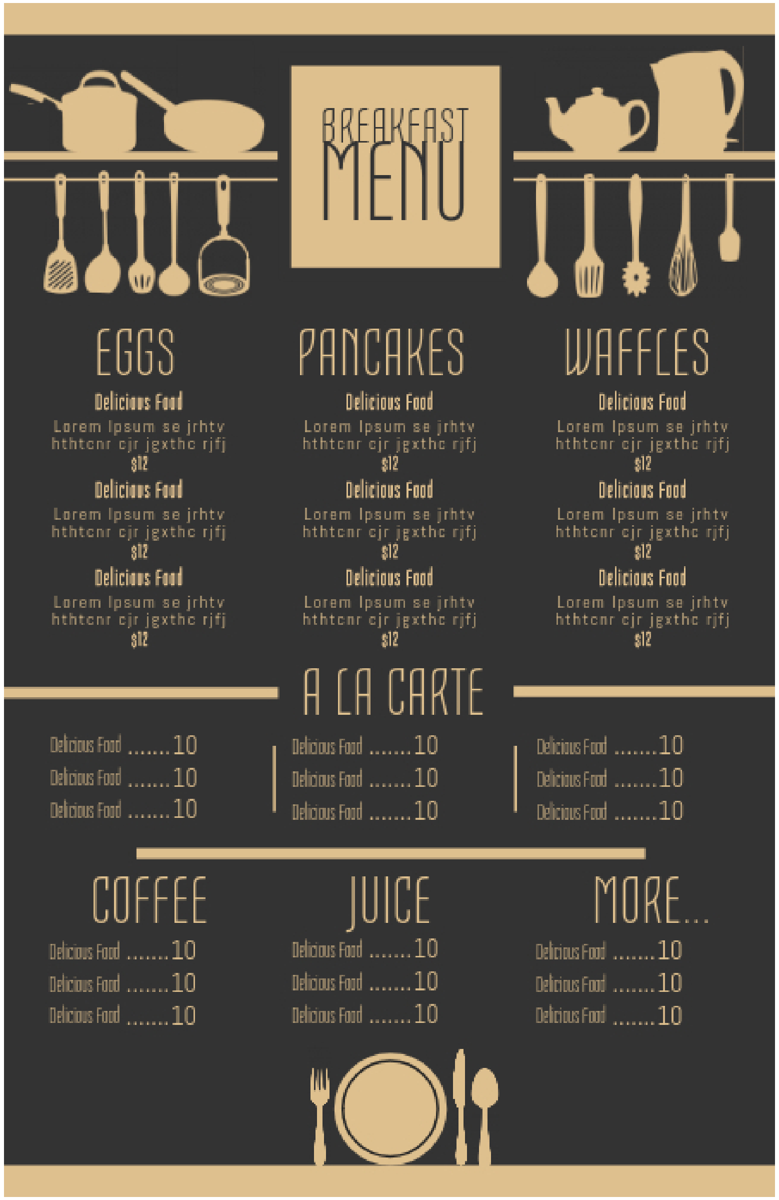
Is this the best model and can we improve it?

The pitfalls of mathematical modelling

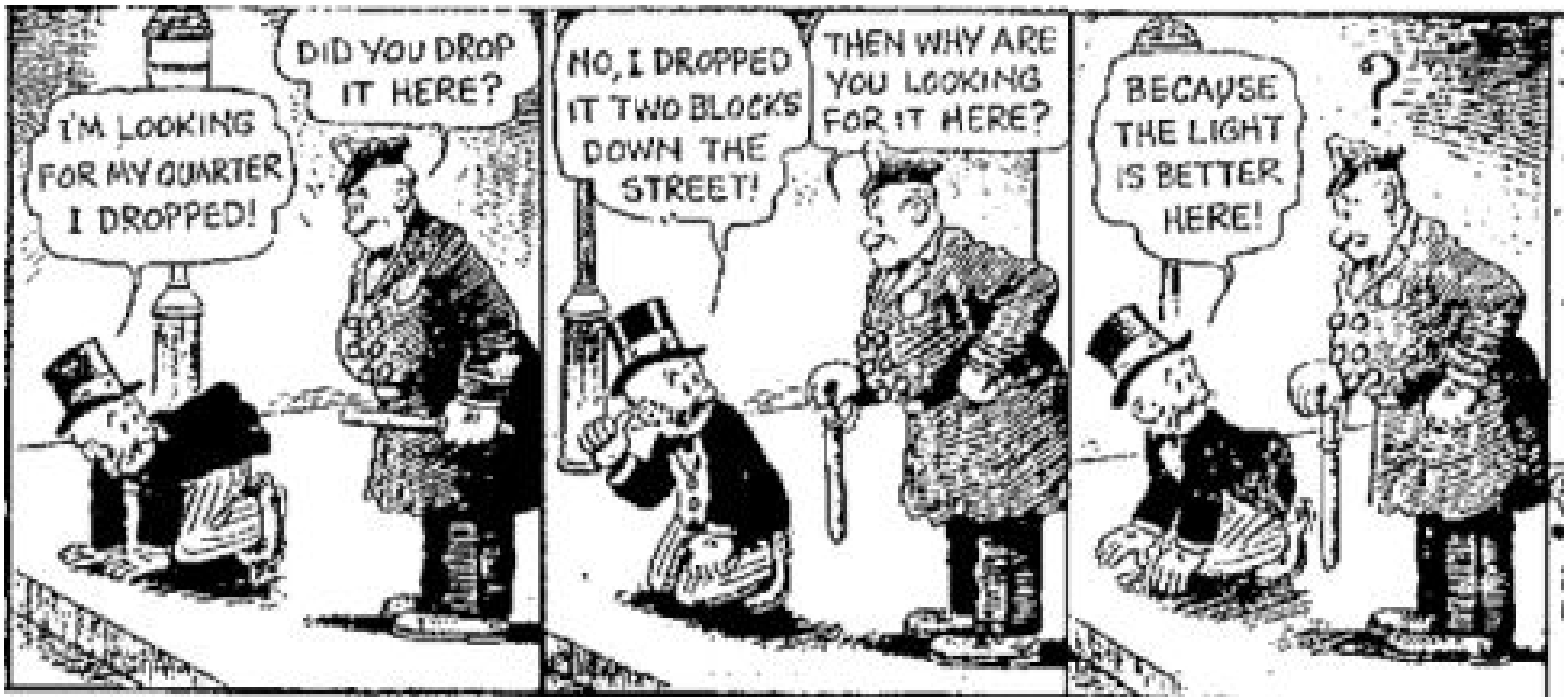
All models are wrong, but some of them are useful

George Box, 1976

Don't eat the menu



The Mathematical Drunkard



The Ivory Tower

Nature will throw out mighty problems, but they will never reach the mathematician.
He may sit in his ivory tower waiting for the enemy with an arsenal of guns, but the enemy will never come to him. Nature does not offer her problems ready formulated. They must be dug up with pick and shovel, and he who will not soil his hand will never see them'

J. Synge. American Mathematical Monthly, 1944

The curse of the formula

$$K = \frac{F(T+C)-L}{S}$$

The formula for the perfect kiss!

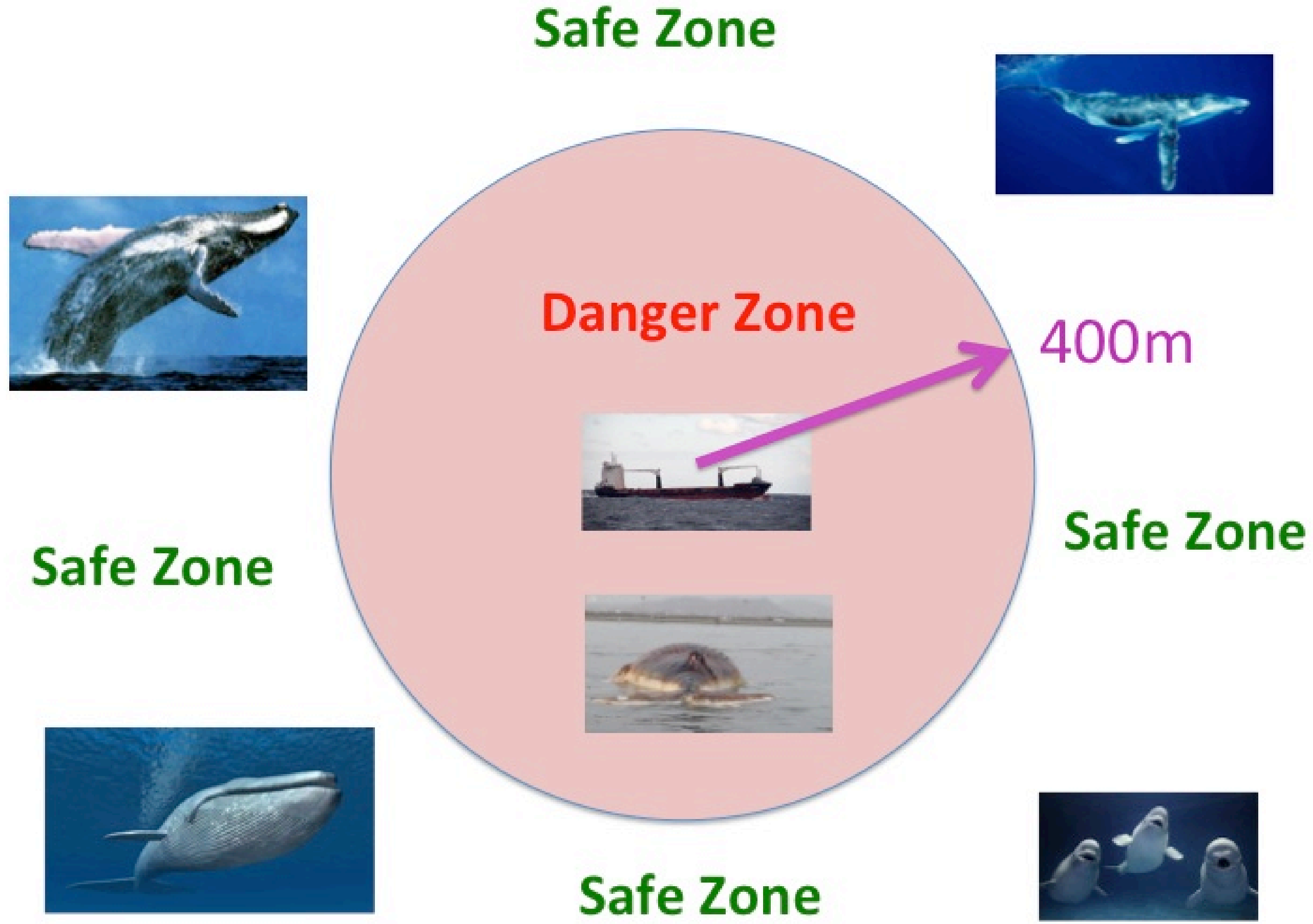
(Don’t forget your brackets!!!)

Teaching Mathematical Modelling:
The Study Group Model

Maths saves the whales

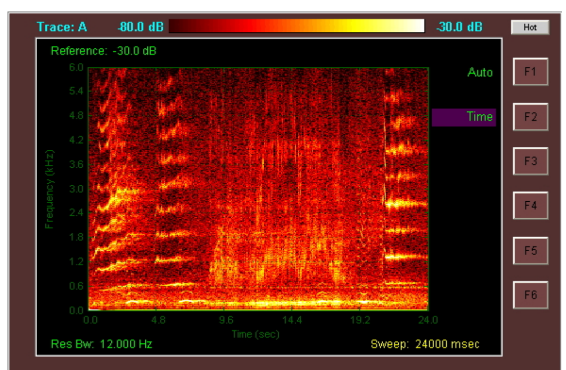
Despite the stopping of hunting, whales are still under threat

Major threat from ship strikes

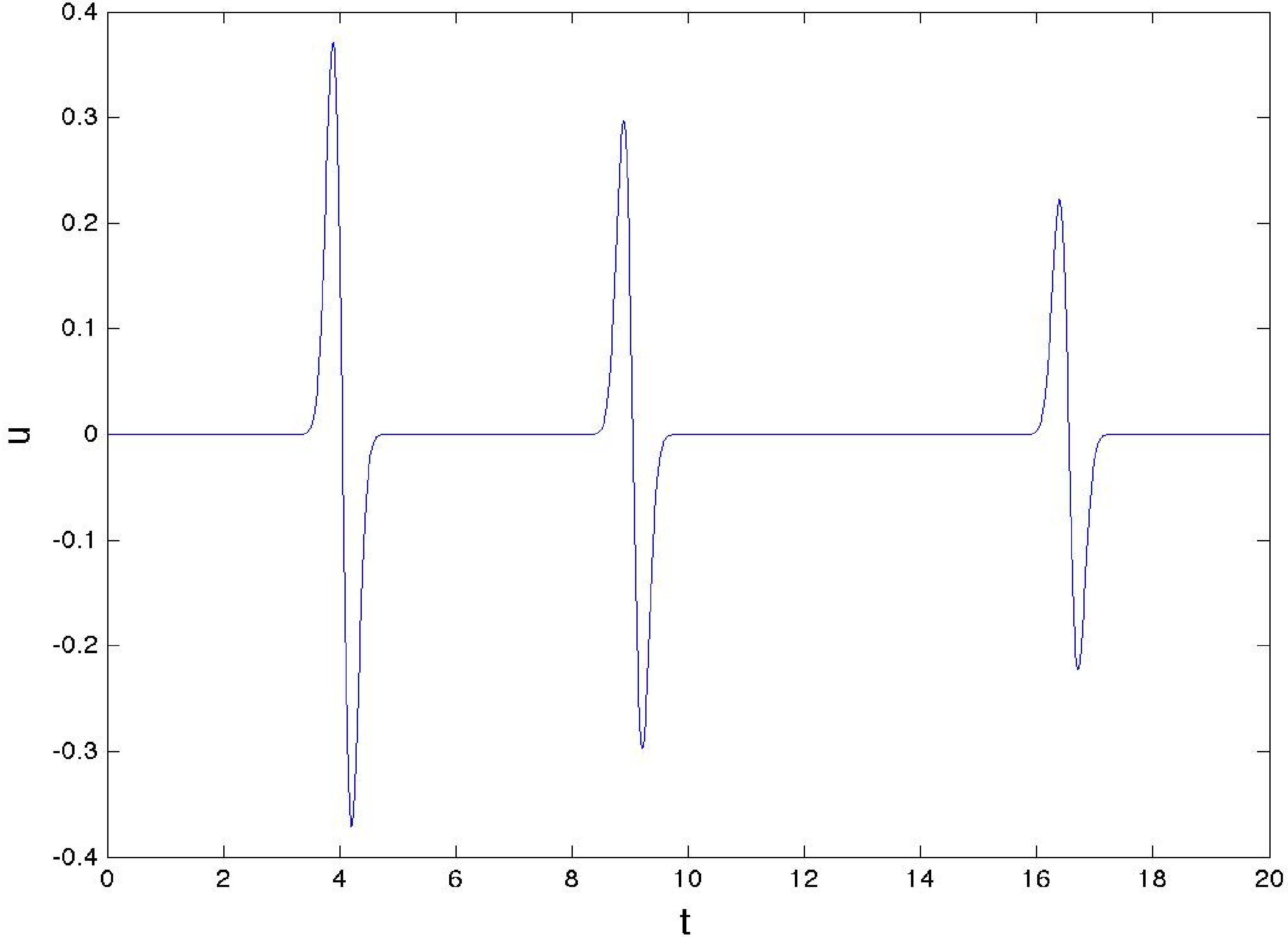

Safe Zone
Danger Zone
400m
Safe Zone
Safe Zone
Safe Zone

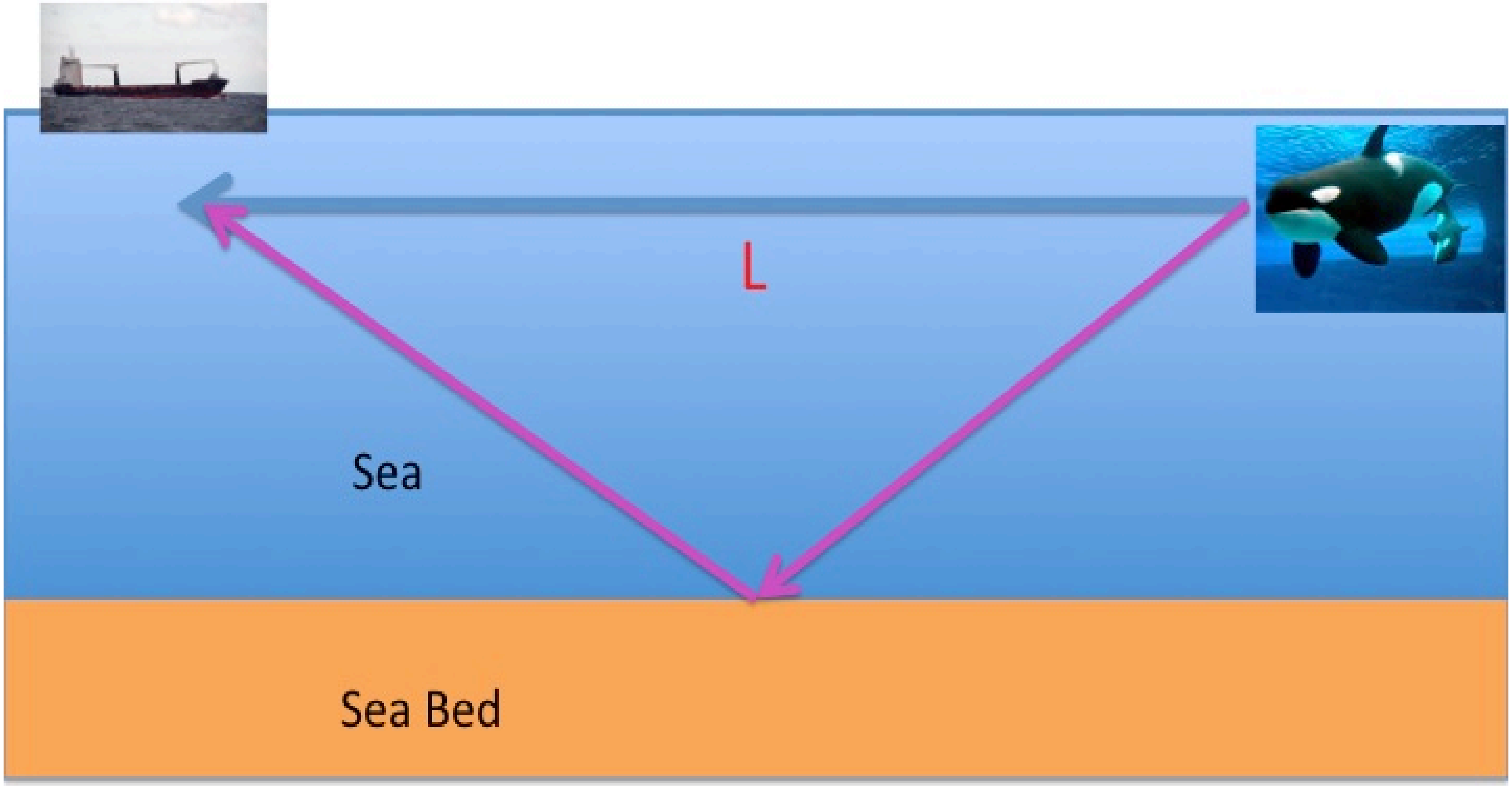
Need to work out how far away the whale is:

- Radar
- Sonar
- Looking
- Listening



Sounds recorded from the whale





Hear **Two Sounds**. A direct one and an echo from the sea bed

We can measure the time difference between them.

The mathematical model

Sounds are pressure waves in water that satisfy the wave equation

$$u_{tt} = c^2 \nabla^2 u$$

Single frequency assumption

$$u(x,t) = e^{i\omega t} w(x)$$

$$\nabla^2 w + \frac{\omega^2}{c^2} w = 0.$$

Scale to a length scale of; X = 200m

$$\nabla^2 w + k^2\ w = 0. \qquad k = \frac{X\omega}{c}$$

Mathematical theorem:

If k is large then the solutions of this equation travel in straight lines

For our problem: f = 30 Hz, $\omega = 2\pi f,$ c = 1500

$$k = \frac{200 * 2\pi * 30}{1500} \approx 25.$$

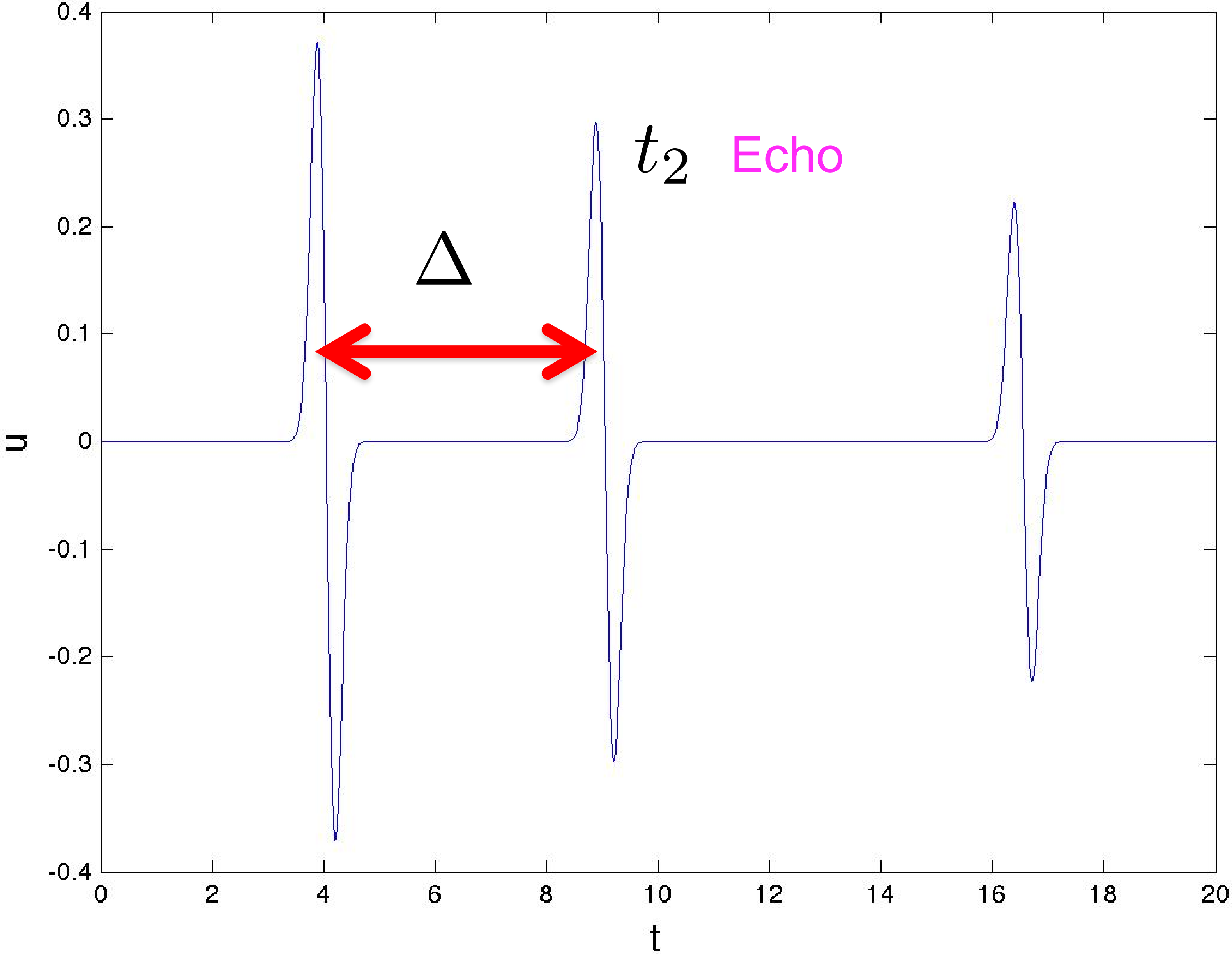
Large enough

If the whale is a distance L away from the boat then the time of arrival of a direct sound from the whale to the boat is given by

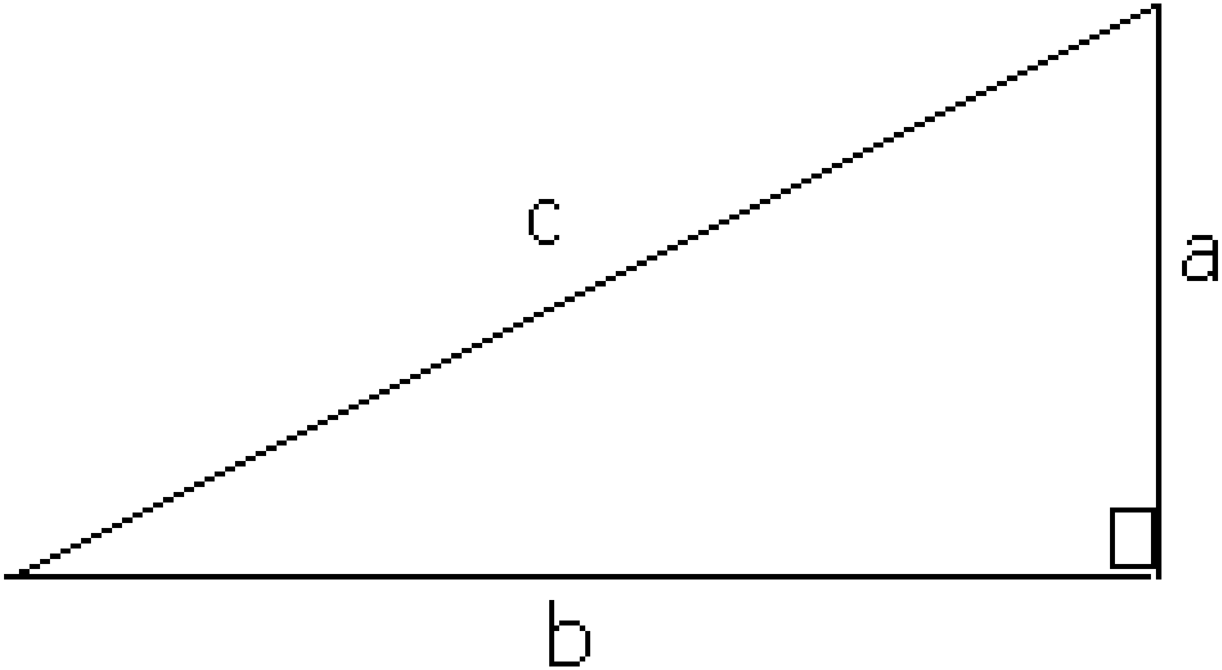
$$t_1 = \frac{L}{c}$$

But we don't know when the sound was emitted. To do this we need to find the time when the echo arrives

$t_1 = L/C$ Direct

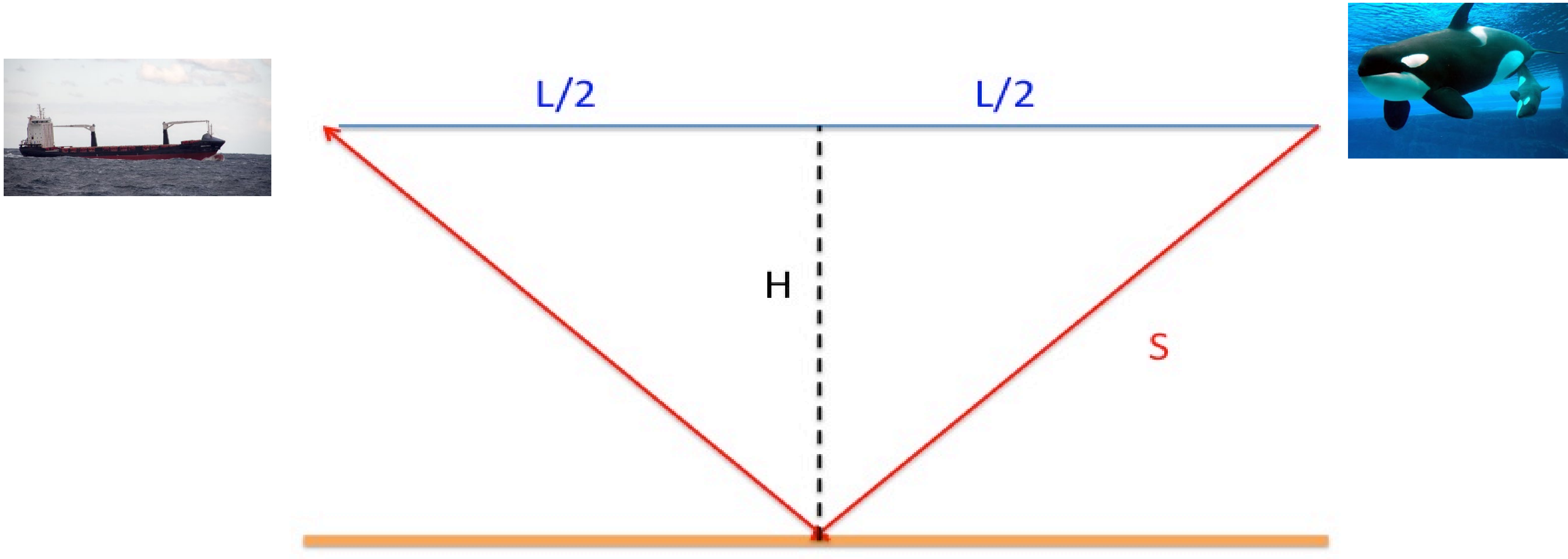


Find the time using Pythagoras' Theorem



$$a^2 + b^2 = c^2$$

A right-angled triangle opined
My hypotenuse squared is refined
For if any one cares
It's the sum of the squares
Of my other two sides when combined!



$$S^2 = H^2 + \frac{L^2}{4}$$

Pythagoras' Theorem implies

$$S = \sqrt{H^2 + \frac{L^2}{4}}$$

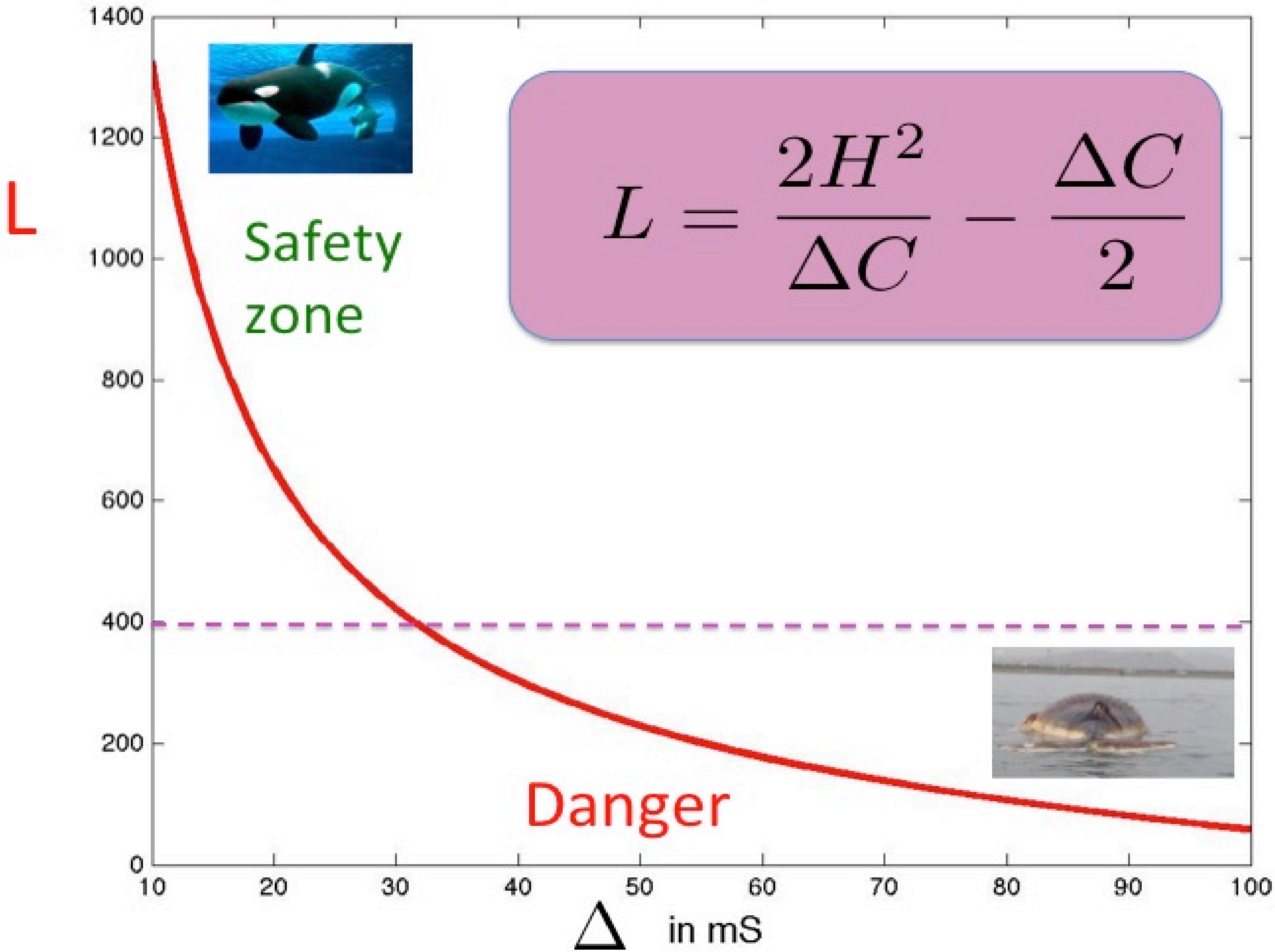
$$2S = 2\sqrt{H^2 + \frac{L^2}{4}} = \sqrt{4H^2 + L^2}$$

$$t_2 = \frac{\sqrt{4H^2 + L^2}}{C}$$

$$\Delta = \frac{\sqrt{4H^2 + L^2}}{C} - \frac{L}{C}$$

$$\Delta^2 + 2\Delta\frac{L}{C} + \frac{L^2}{C^2} = \frac{4H^2 + L^2}{C^2}$$

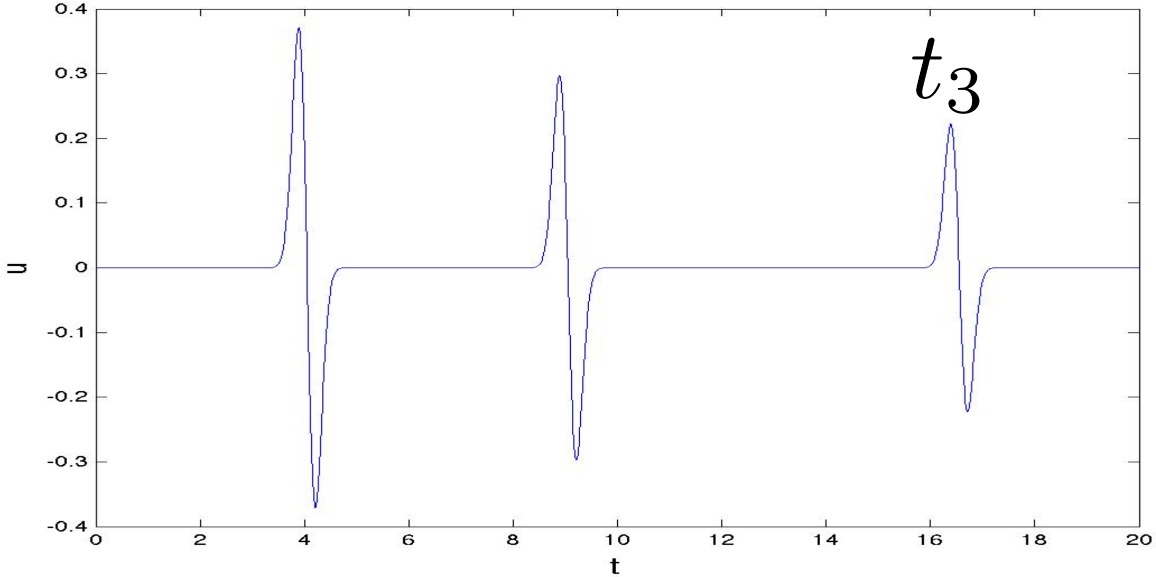
$$L = \frac{2H^2}{\Delta C} - \frac{\Delta C}{2}$$


L
Safety zone
$$L = \frac{2H^2}{\Delta C} - \frac{\Delta C}{2}$$
Danger
0
200
400
600
800
1000
1200
1400
10
20
30
40
50
60
70
80
90
100
Δ in mS

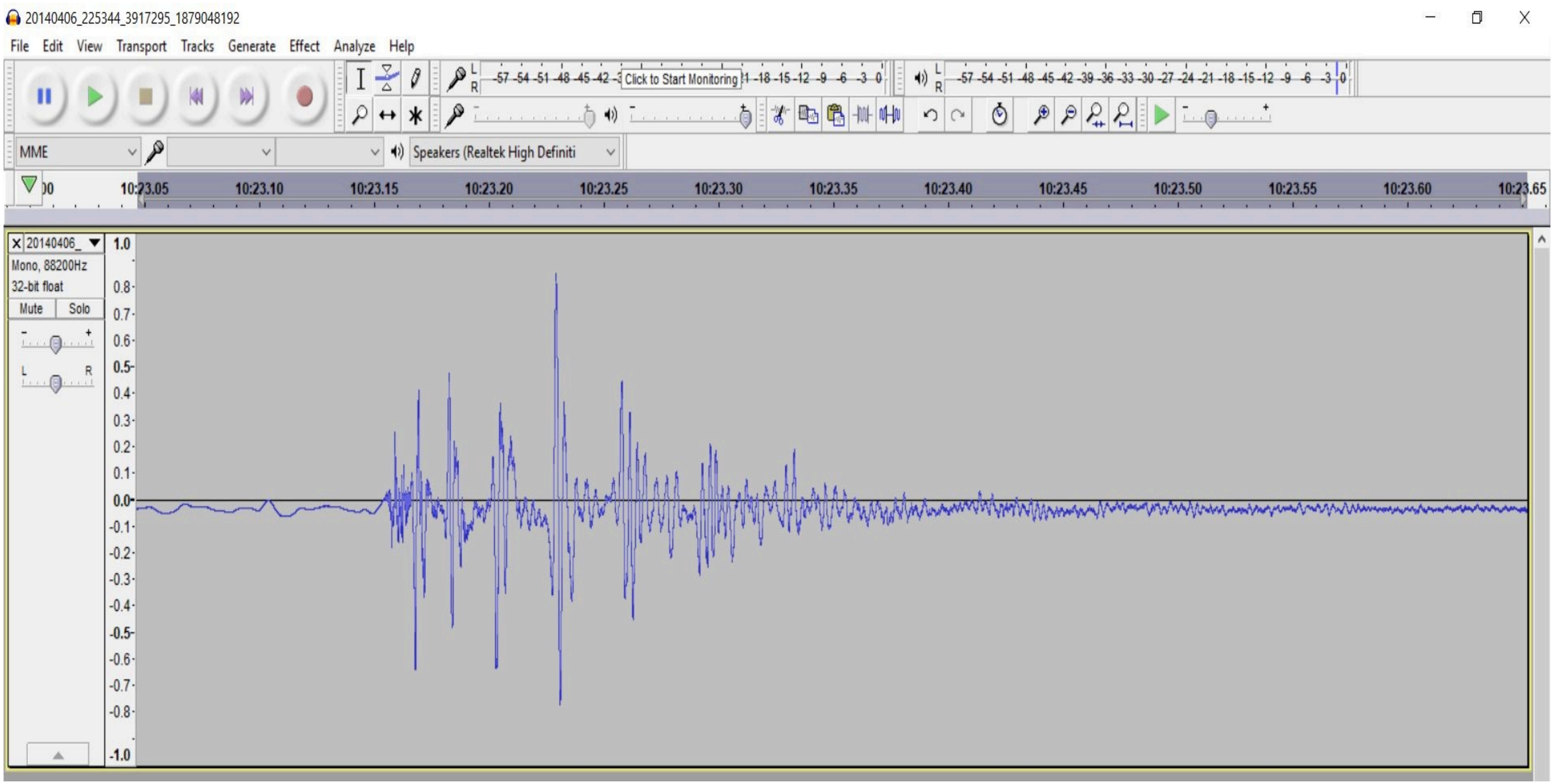
Problems with the model and how we can improve it

1. We don't know the depth H

Overcome this by making measurements of many echoes.
These give H and much more besides.

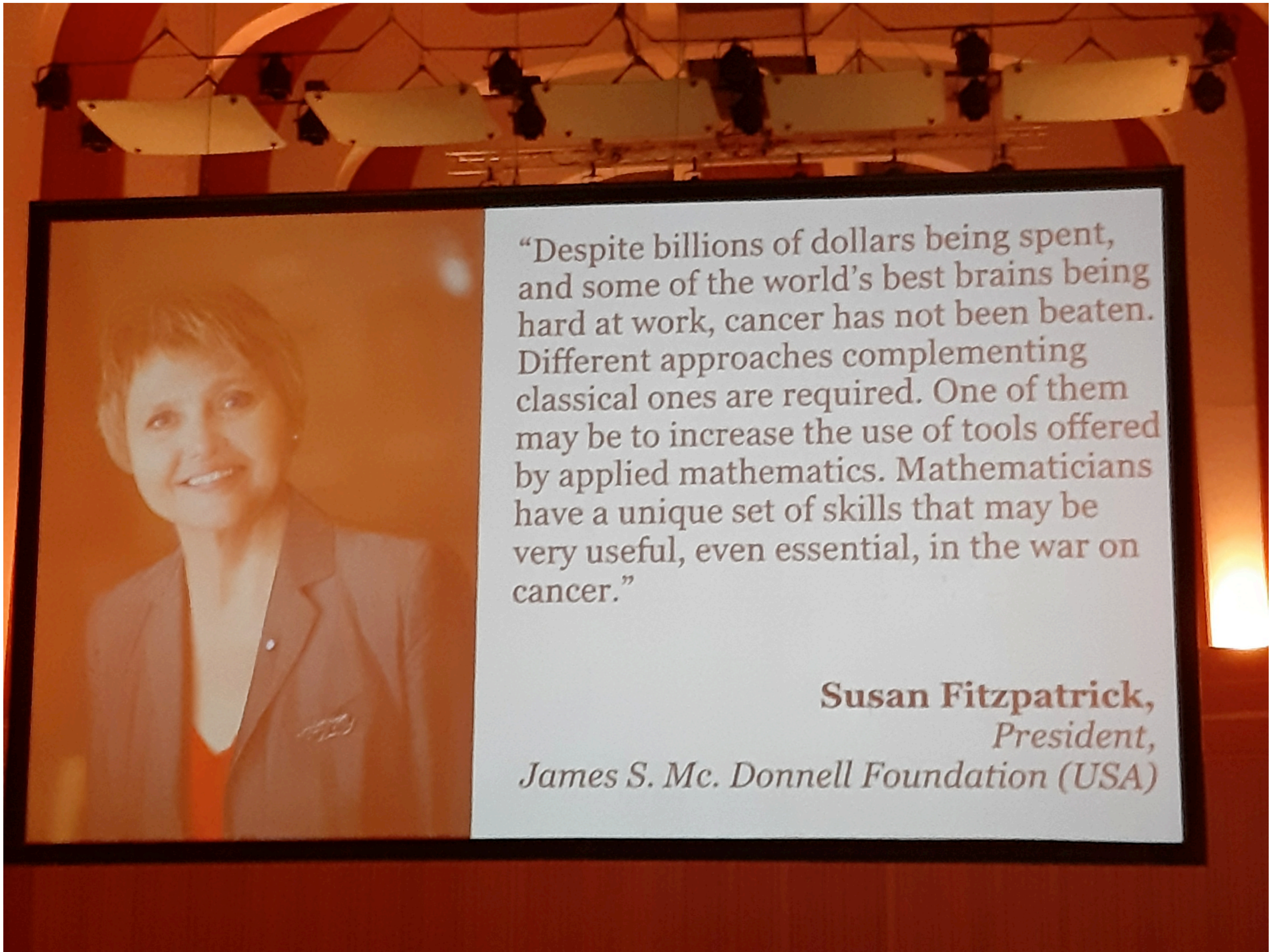


2. Nature is really a bit wavey and quite noisy



Overcome by putting in a small correction to the straight line model

Can maths help to cure cancer?



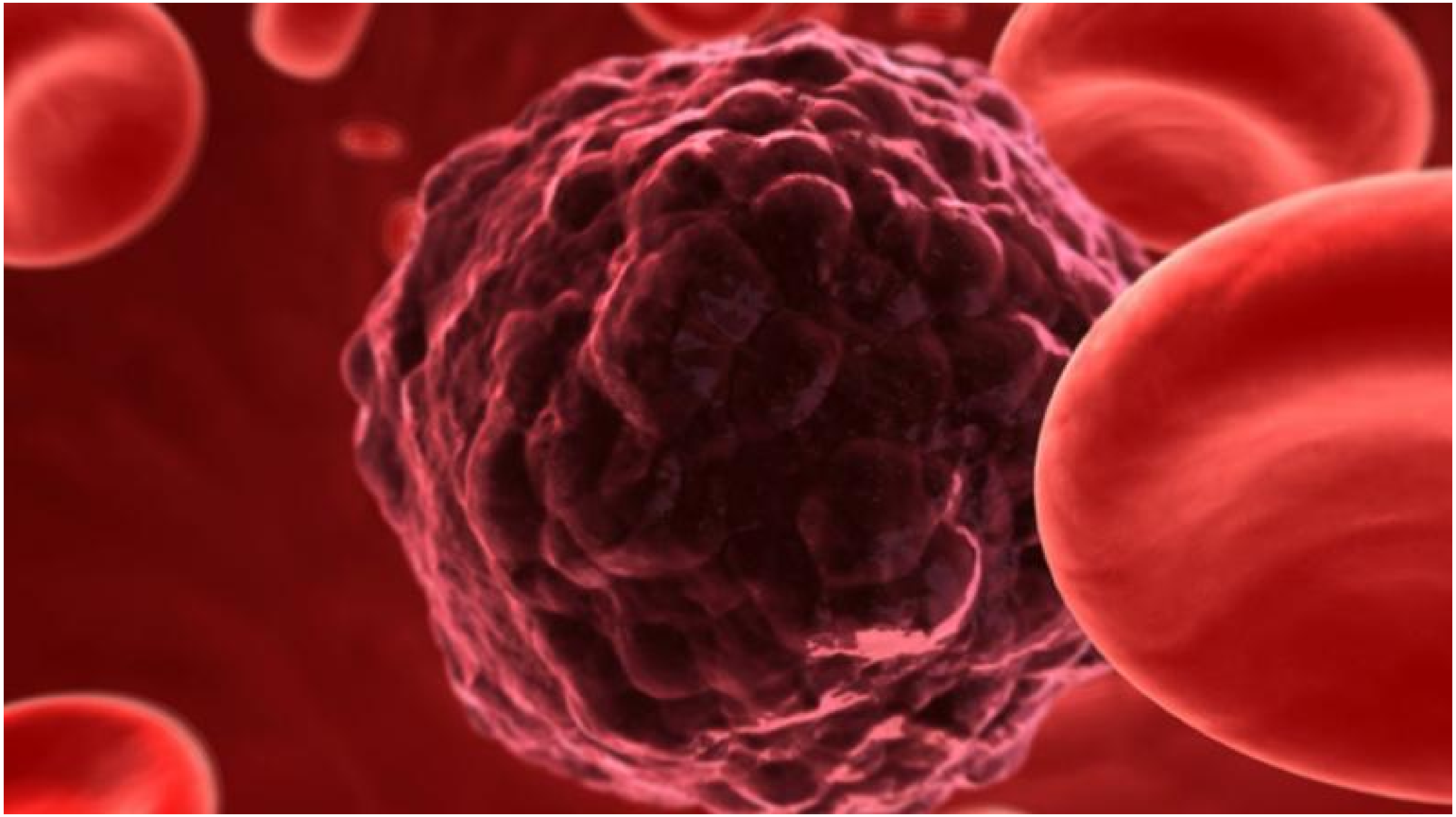
1. Tumour growth and treatment

Cancers are diseases characterised by rapid, uncontrolled cell growth resulting in the formation of a tumour

Mathematical and computational modeling approaches, usually based on differential equations, have been applied to every aspect of tumour growth

Mathematical models allow us to identify the most effective drug combinations for cancer patients. They are also deepening our understanding of how and why cancer cells often become resistant to chemotherapy drugs

A model for tumour growth

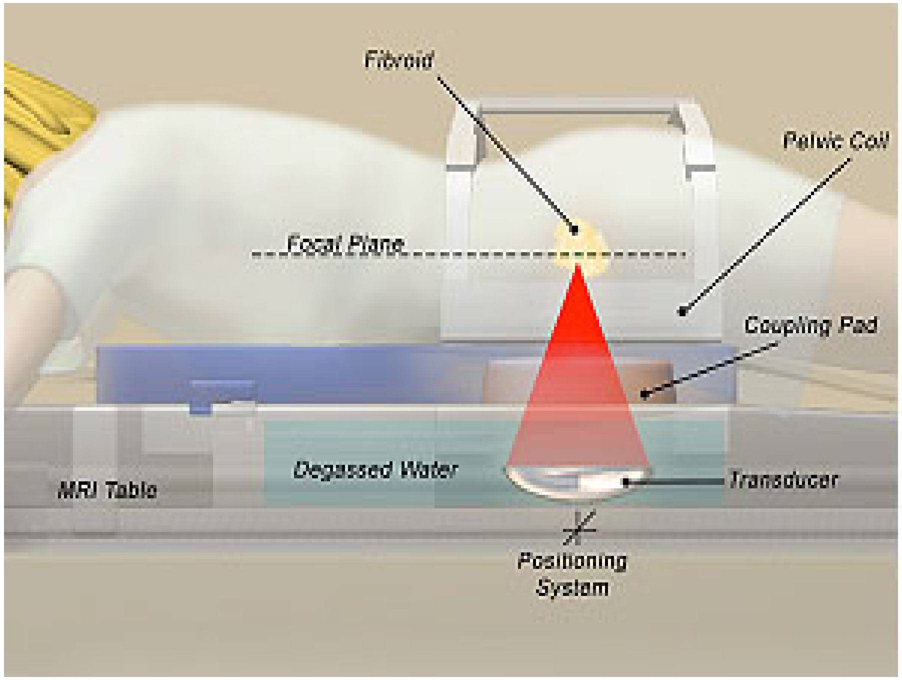


$$c_t = \nabla.(D\,\nabla c) + \rho\, c\,(1 - c/K)$$

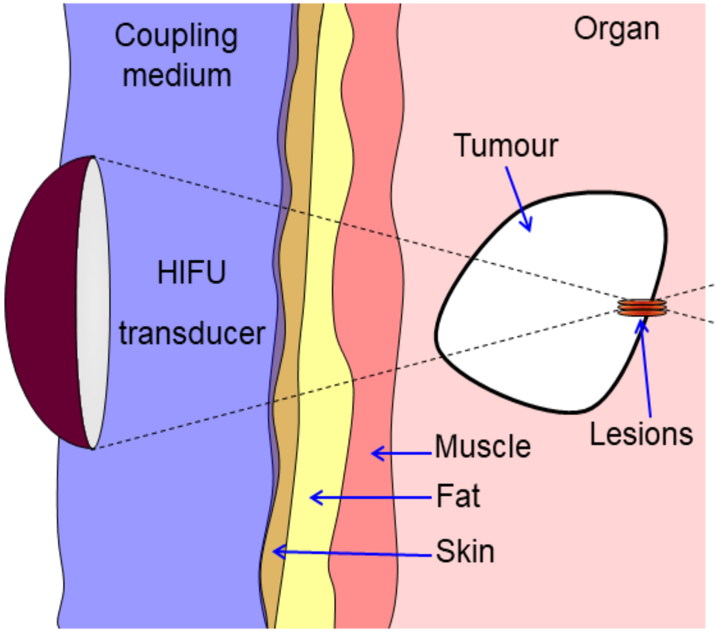
2. Destroying tumours using sound

High intensity ultra-sound can be focused onto a tumour to destroy it

Process is monitored using MRI: MR-HIFU



Whole process works due to mathematical modelling



- Intensity and timing of the sound
- Focusing of the sound waves
- Imaging the result

Model: Wave equation very similar to whale spotting

Frequency f = 1 MHz, Length scale X = 5 mm

$$k = \frac{2\pi * 10^6 * 5 * 10^{-3}}{1500} = 20.9$$

Very similar to whale value!

Can model sound path as a straight line and calculate its intensity Q

Can then find the temperature T from the equation

$$\rho c\, T_t = \nabla . \left(\kappa(x) \nabla T \right) - \omega c \left(T - T_b \right) + Q(x, t)$$

Conduction

Blood

Sound

And then work out the physiological damage

Technique works and is undergoing trials

North American first in children: SickKids doctors destroy bone tumour using incisionless surgery

In Conclusion

The the potential applications of mathematical modelling are almost limitless

Watch this space!

But always be aware of the limitations of any model.