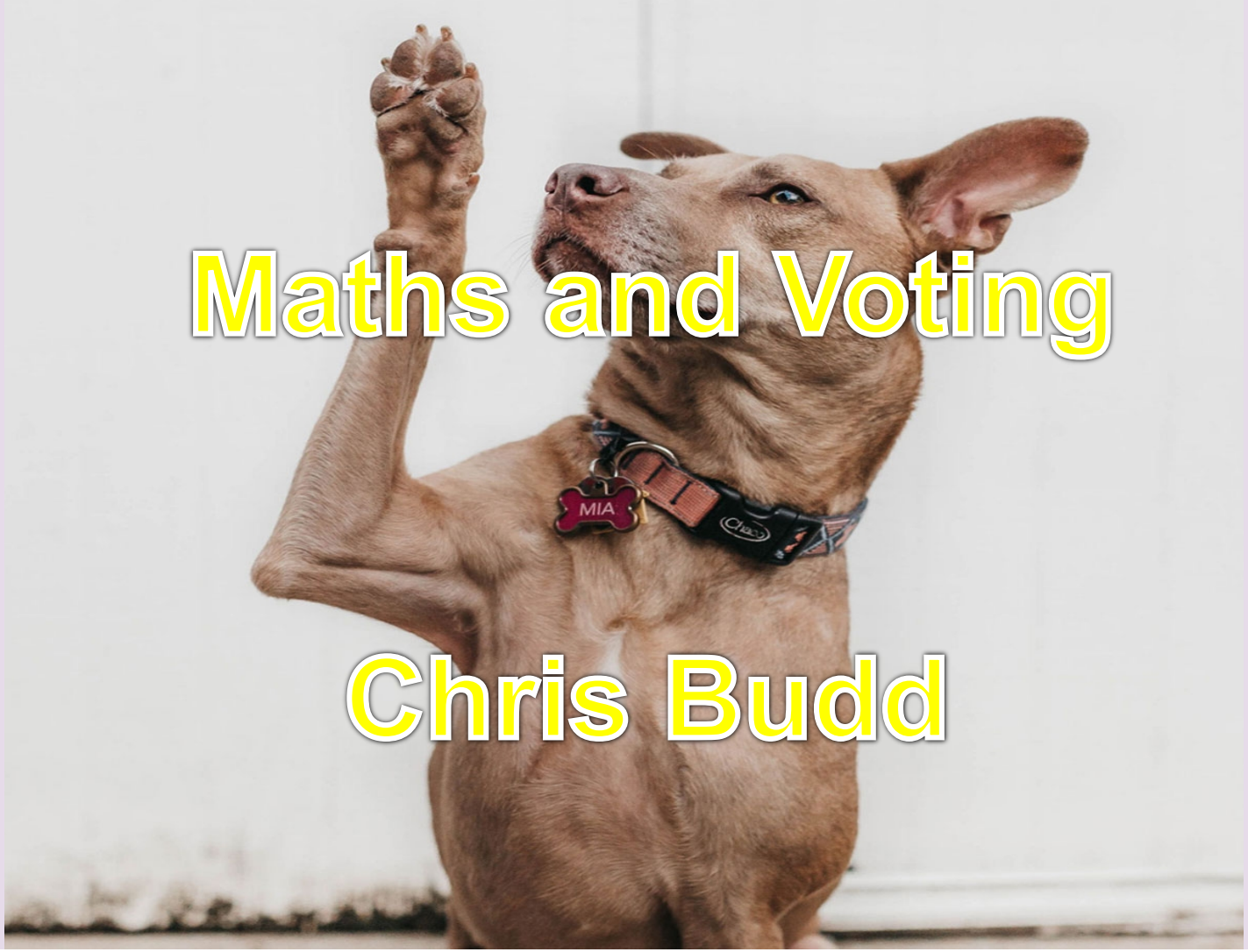
Maths and Voting



Chris Budd

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We live in interesting times!

- General election
- Brexit
- Strictly Come Dancing

All involve major decisions based on voting

Want to turn the opinions of a group into a final decision

BUT …

How can we make sure that this process is fair, representative, and anonymous

A rigorous approach to voting

Kenneth Arrow. 1972 Nobel Prize in Economics

Arrow assumed an election had multiple candidates A,B,C .., and each voter could express a range of opinions on each one

Arrow's Axioms for Fair Voting

1 **(Dictator)** The system should reflect the wishes of more than one voter, so there is no dictator.

2 **(Unanimity)** If all voters prefer candidate A to candidate B then A should come out ahead of B in the final vote.

3 **(Universality)** The voting system should always return one clear result.

4 **(Independence of irrelevant alternatives)** In the final result, the ranking of A above B should only depend on how individual votes ranked A compared to B and not how they ranked them when compared to a third alternative C. No tactical voting

Some more reasonable conditions for fair voting

Majority condition: A candidate who is top choice for a majority of the voters should get elected.

Monotonicity: It should neither be possible to prevent the election of a candidate by ranking them higher on some of the ballots, nor possible to elect an otherwise unelected candidate by ranking them lower on some of the ballots.

Anonymity: it should not be possible to tell from the vote, who voted for which candidate.

Practicality: The method should be polynomial in time

Problem!

There's NO voting system which satisfies them all

All voting systems have flaws and are compromises between different priorities

Example: The Borda voting method

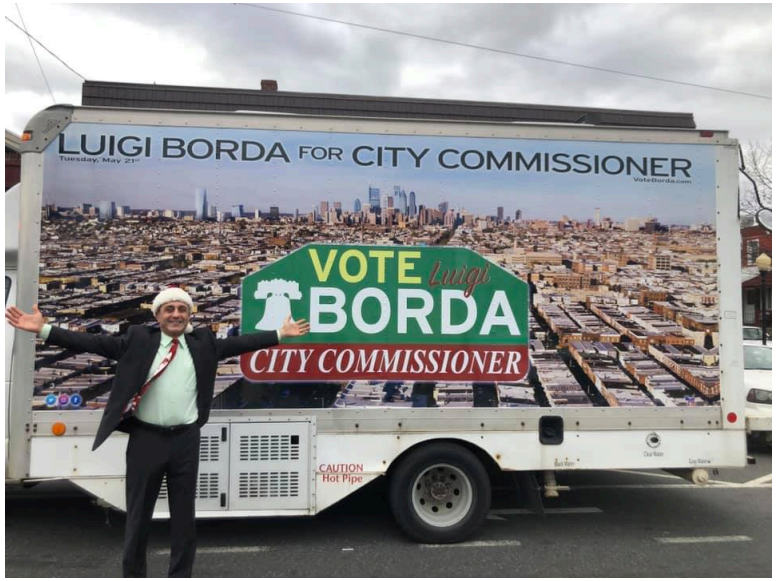
Election with N candidates eg. Recruiting a lecturer

Purest form:

Each voter gives each candidate a mark from N-1 to 0

Votes are added

Highest total vote wins



Strictly Come Dancing:

Each judge gives each dance pair a mark from 10 to 1
Marks added to give a ranking. Combined with audience rank

First Borda example:

Candidates

	A	B	C
V1	2	1	0
V2	2	0	1
V3	1	0	2
Total	5	1	3

A wins with 5 votes

Second Borda example:

	A	B	C
V1	2	1	0
V2	0	2	1
V3	2	1	0
Total	4	4	1

Result is a tie

How well does the Borda method match Arrow's conditions?

Dictator The Borda system is a consensus system electing broadly supported candidates. However, if there are only a few voters it is possible for one to be a dictator by giving an otherwise popular candidate a very low score or vice versa.

Unanimity and *Monotonicity* are satisfied. If all voters prefer A to B then A will always get a higher ranking than B. It follows the sum of the rankings of A will be higher than the sum of the rankings of B.

Or mathematically

if $X > Y$ and $W > Z$ then $X + W > Y + Z$.

Universality fails. We have seen this in the election above where A and B tie for first place. Rerunning the vote may well sort this out, but this cannot be guaranteed.

Independence also fails. This is more subtle, and is a weakness of the Borda system which can be exploited in tactical voting

Linked to this

The Borda method
fails the majority condition!!

Third Borda example:

	A	B	C	D
V1	3	2	1	0
V2	3	2	1	0
V3	3	2	1	0
V4	0	2	3	0
V5	0	2	0	3
Total	9	10	6	3

Candidate B wins the election.

But …

A has won the majority of the voter preferences

B has never come top with any voter

A could have lost due to tactical voting by V4 and V5

One pair gets votes: 9 9 9 1 = 28

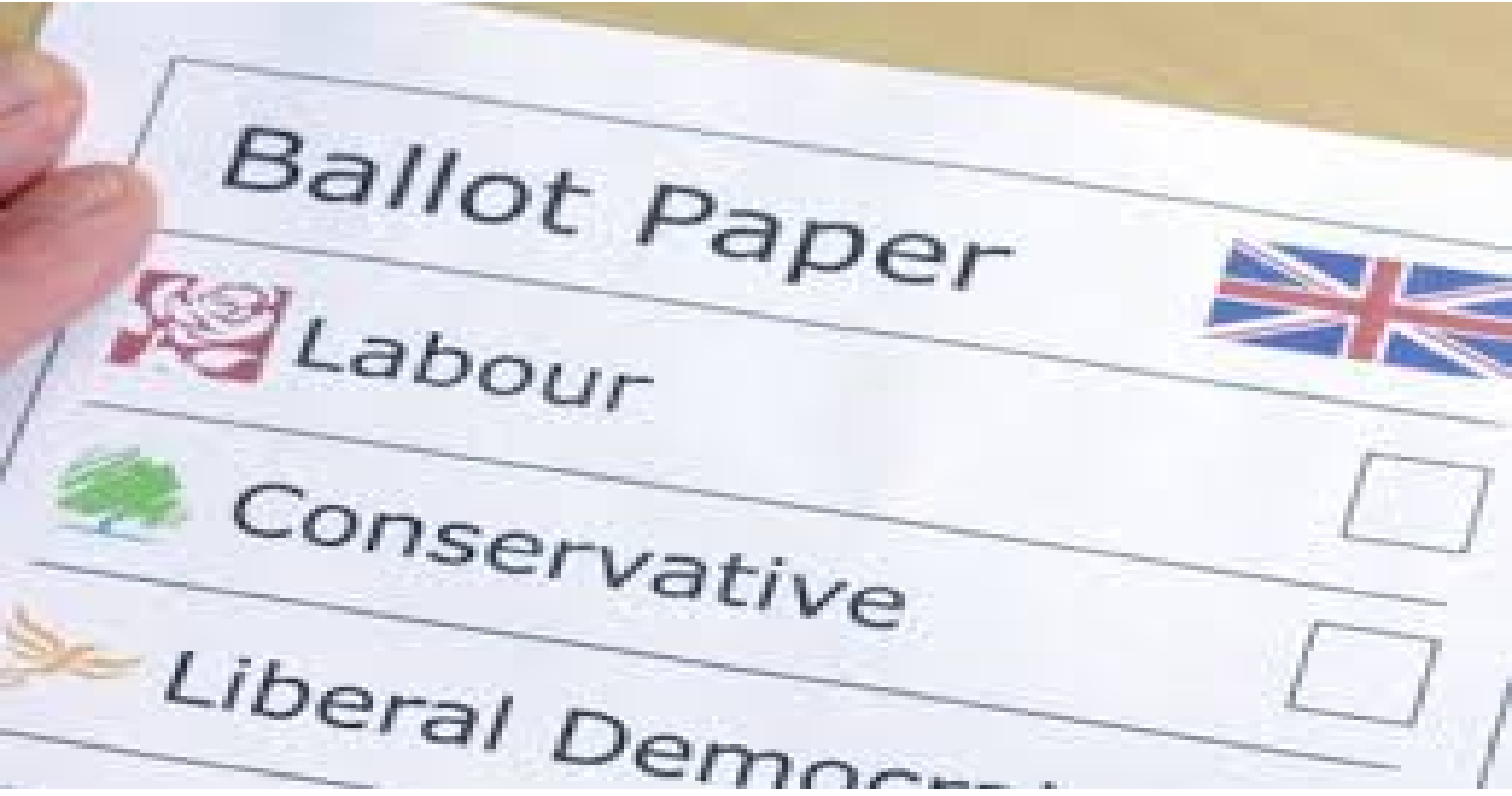
Another gets votes: 8 8 8 8 = 32

Second pair wins despite first pair being preferred by 3 judges

The last judge is a dictator

Commonly used voting methods

First Past the Post (FPTP)



Single constituency or single issue

Voters express a single preference for a candidate or issue

The one with the most votes wins

Eg. Parliamentary votes, Brexit

Political equations

Sir, Norman Sanders (letter, Oct 9) asked how many MPs needed to know how to solve simultaneous linear equations. The answer is, hopefully, all of them. For example, working out how many Conservative MPs need to vote against a government motion for the government to lose a majority means solving exactly such equations.

Similar equations need to be solved for an MP to work out how many votes they need to get elected, and more complex equations will have to be solved if we move over to a single transferable vote system.

I expect every MP uses Google to check facts, and Google relies on being able to solve billions of simultaneous linear equations quickly — as does any airline, electricity company, weather centre or global corporation.

Mathematics affects all our lives, and with the increased use of data and algorithms its impact will only increase. That is probably why maths is the most popular A level subject.

PROFESSOR CHRIS BUDD, OBE
Gresham professor of geometry
University of Bath

To work the first past the post you need the mathematics of simultaneous linear equations

Party A: 310 members Party B: 250 members

All of Party B will vote against a motion.
10 members of Party A will abstain

How many of Party A have to vote for the motion for it to carry?

x : votes for y : votes against

$x + y + 10 = 310$ Members of party A

$x = 250 + y + 1$ Votes to carry the motion

$2x + y + 10 = 560 + y + 1$

$2x = 551$ so that $x = 275.5$

276

Advantages: Traditional, clear decision

Disadvantage 1: No allowance for a preference

Disadvantage 2: Split votes

1. Increase Secondary School funding,
2. Increase Primary School funding,
3. Give no extra funding to schools.

Voting: 25% for 1, 35% for 2, and 40% for 3.

Decision: Give no extra funding!

Disadvantage 3: Possibility of error

What happens if not everyone votes?

2016 Brexit result was 52%-48% in favour of Leave

The turnout rate was 72%

37% of the British electorate actually chose to vote Leave.

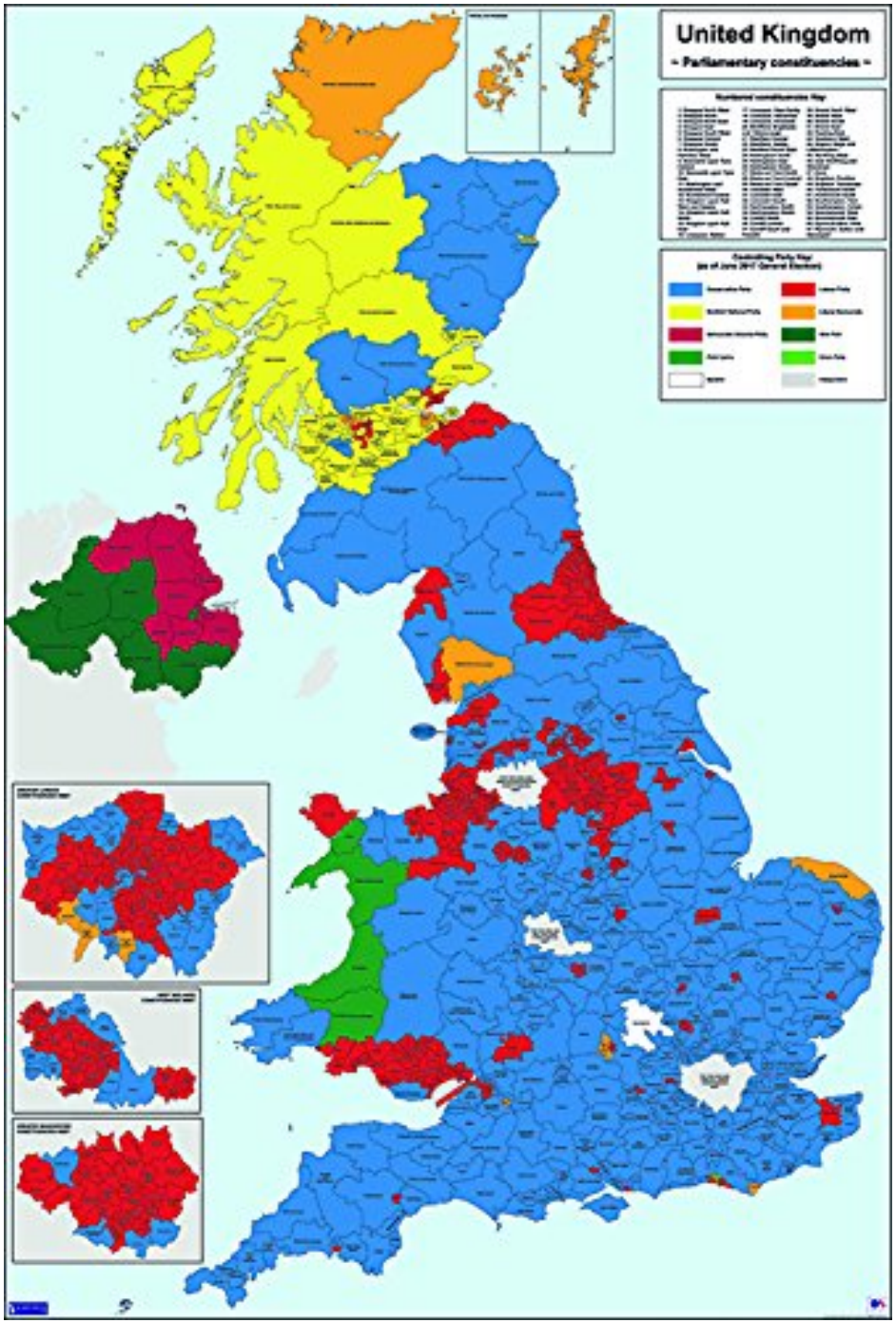
Q. Is a 52% majority on a 72% sample strong enough evidence for a greater than 50% majority from the whole sample?

Subtle question in statistics which is a matter of hot debate!

Uncertainty is a good reason for insisting on a margin of error in referendums such as 60:40

Multiple constituency voting

How to make a bad thing worse!



UK/US General Elections

FPTP run in many constituencies

The party with the most elected members wins

Gives very distorted results

Eg. Recent US election

Donald Trump and the Republicans
got 304 electoral votes and 46.1% of the vote

Hilary Clinton and the Democrats got
227 electoral votes and 48.2% of the vote

Example: Opposite to Borda voting

	A	B	C
Constituency 1	10,000	9,000	1,000
Constituency 2	10,000	9,000	1,000
Constituency 3	1,000	9,000	10,000

- A wins the election
- B comes second in all constituencies and wins no seats
- B gets the most votes

Gerrymandering

Governor of Massachusetts, Elbridge Gerry approved of an oddly-shaped voting district with the explicit outcome of trying to rig the election

It worked!!

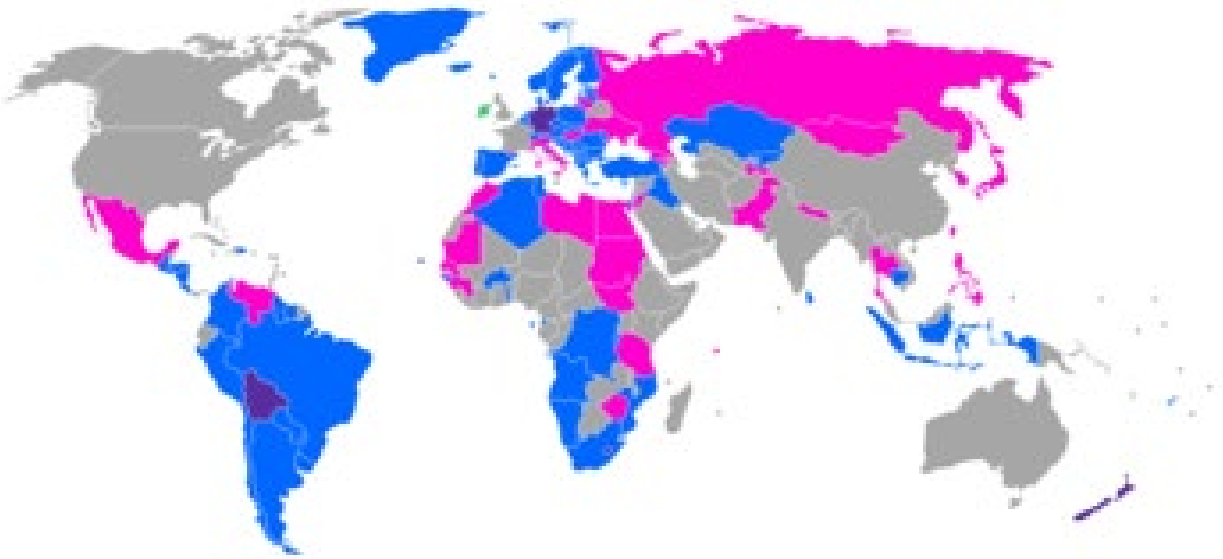
Practice still goes on today!

Proportional Representation

Avoids many of these problems

IDEA: Divide N seats in proportion to the votes received from the voters

Widely used around the world eg. EU Parliament



Main Problem 1

You end up with a fractional number of seats

Eg. 11 seats, with votes in proportion

1/2 : 1/3: 1/6

Gives seats allocated as:

5.5 3.66666 1.83333333

Main Problem 2

Parties/countries with a small number of votes may get no seats at all in a strict proportional election

Addressed in the European elections where following 2009 each country has to have between 6 and 96 seats

Smaller countries over represented *degressive representation*

The d'Hondt method for seat allocation in PR

Parties 'buy seats' with their votes until no seats are left
Seat price starts high, and is reduced as seats are allocated

.

Operation:

- The party with the most votes wins **one seat**
- The votes V for each party are divided by the number of seats s it has plus one to give

$$N = V/(s + 1)$$

- Second seat is given to the party with the largest value of N

- Process continues until all of the seats have been allocated

Used in the European elections on May 22, 2014

Example: 7 seats to allocate to 4 parties A,B,C,D with votes

A = 100 000, B = 80 000, C = 30 000, D = 20 000

1:	A gets one seat	
2:	N(A) = 50 000, N(B) = 80 000, N(C) = 30 000, N(D) = 20 000	B gets one seat
3:	N(A) = 50 000, N(B) = 40 000, N(C) = 30 000, N(D) = 20 000	A gets one seat
4:	N(A) = 33 333, N(B) = 40 000, N(C) = 30 000, N(D) = 20 000	B gets one seat
5:	N(A) = 33 333, N(B) = 26 666, N(C) = 30 000, N(D) = 20 000	A gets one seat
6:	N(A) = 25 000, N(B) = 26 666, N(C) = 30 000, N(D) = 20 000	C gets one seat
7:	N(A) = 25 000, N(B) = 26 666, N(C) = 15 000, N(D) = 20 000	B gets one seat

The seat allocation is **A:3, B:3, C:1, D:0**

The method tends to favour larger parties

Mathematically sophisticated voting

Many mathematicians have considered how to produce ‘optimal’ voting strategies

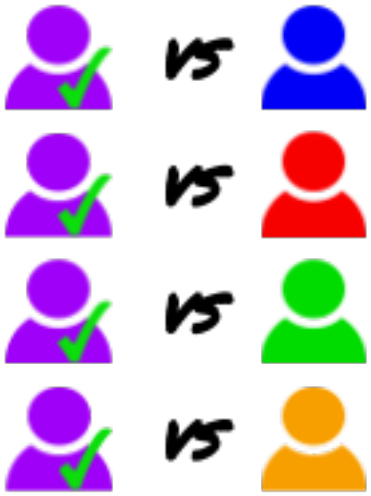
18th-century French mathematician Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet (1743-1794)

‘Gold standard for voting’

In a pure Condorcet method the choices of each voter are compared against everyone else in a series of tournaments.

If one candidate wins all of the tournaments then they win over all

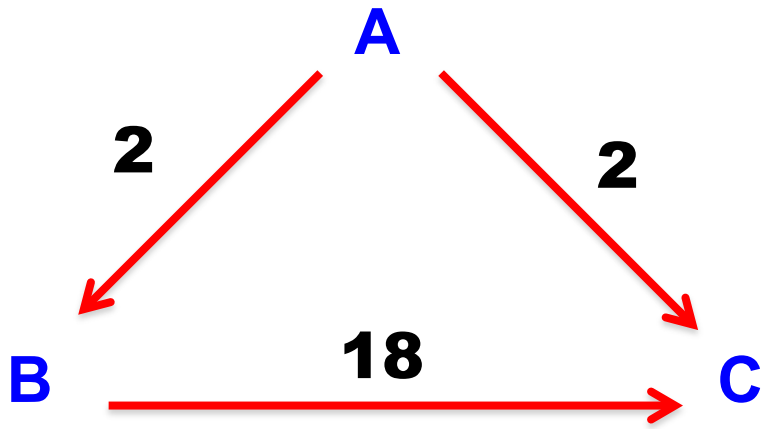
They become the Condorcet Winner



Condorcet winners don't always exist, but if they do we would want our voting method to select them

Example 1: A clear winner

Number of voters	Preferences
10	A > B > C
1	A > C > B
5	C > A > B
0	C > B > A
9	B > C > A
5	B > A > C

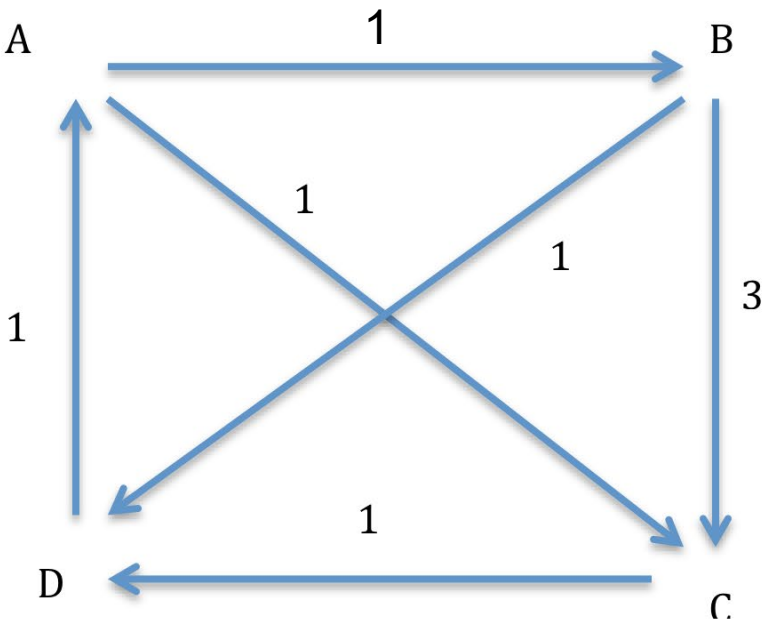


A is the Condorcet winner (B is the Borda winner)

Example 2: A clear tie

Example 3: Not quite a winner

Number of votes	Preferences
3	A > B > C > D
1	D > B > A > C
1	D > C > A > B
1	C > D > B > A
1	B > D > C > A



- No Condorcet winner
- FPTP: A
- Borda: B

Who has won?

Copeland's method

Elect the candidate who wins the most head to head contests

In the example, A and B tie for first place in Copeland's method

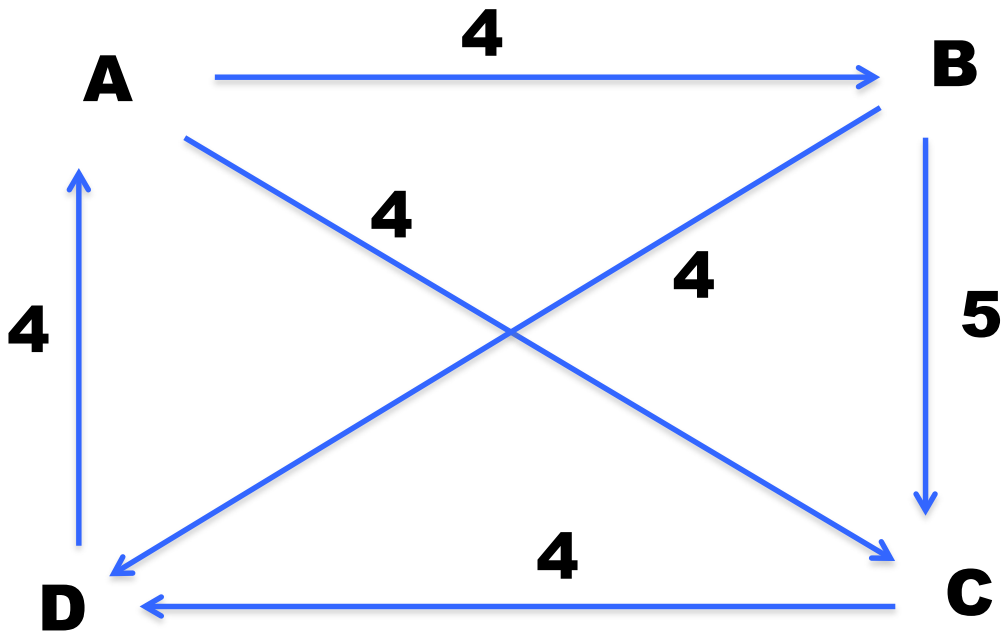
This is not unusual and means that the method is not widely used in practice

A version is used in Premier League football

Shulze's method

A complicated but very fair method. Now used for internet voting eg. Debian and Wikimedia

Draw the same graph as before but show how much each candidate beats the other by



Construct paths from candidate X to candidate Y

Strength P(X,Y) of path is value of the ‘weakest link’

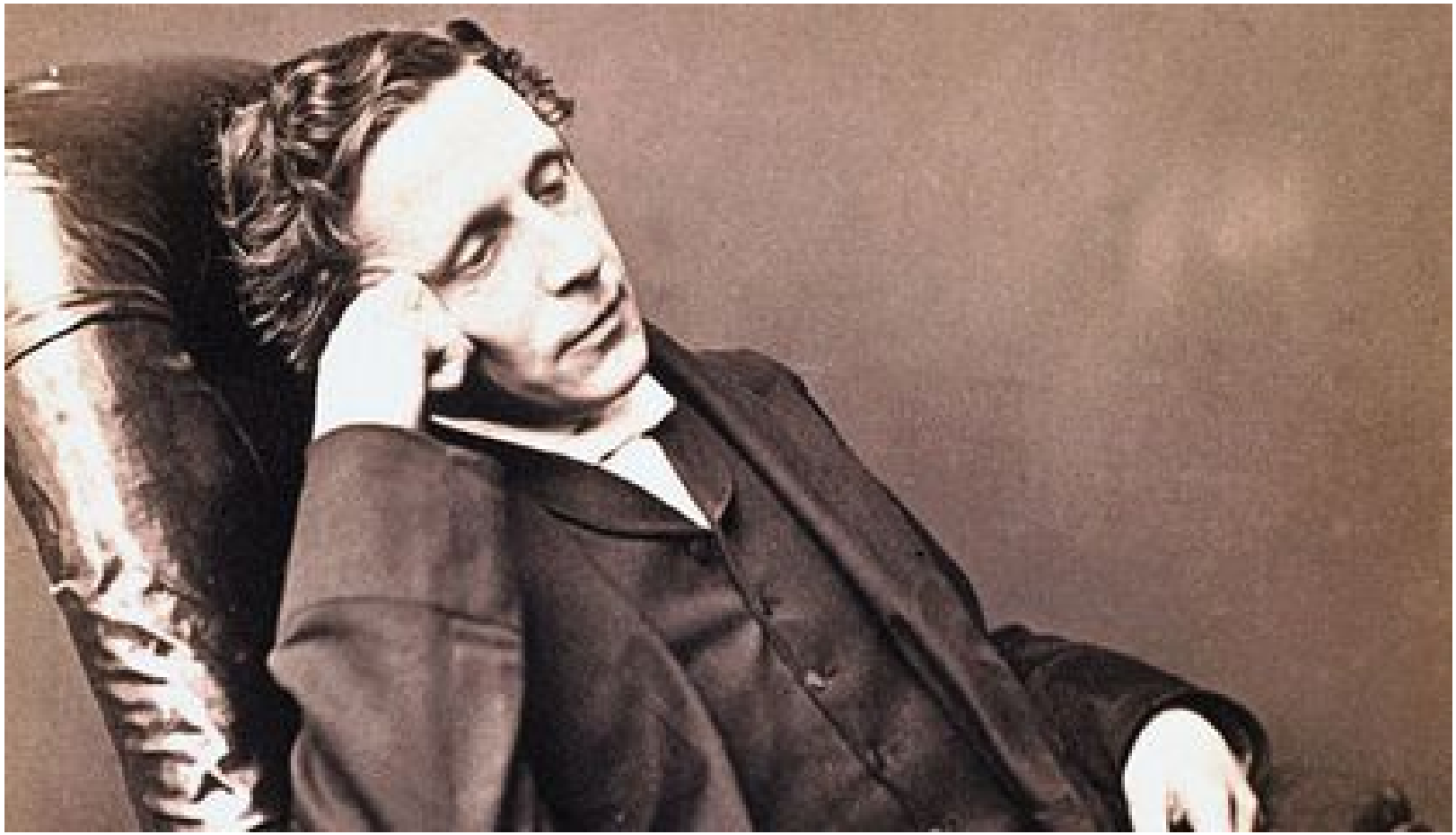
	Path to A	B	C	D
Path from A	x	4	4	4
B	4	x	5	4
C	4	4	x	4
D	4	4	4	x

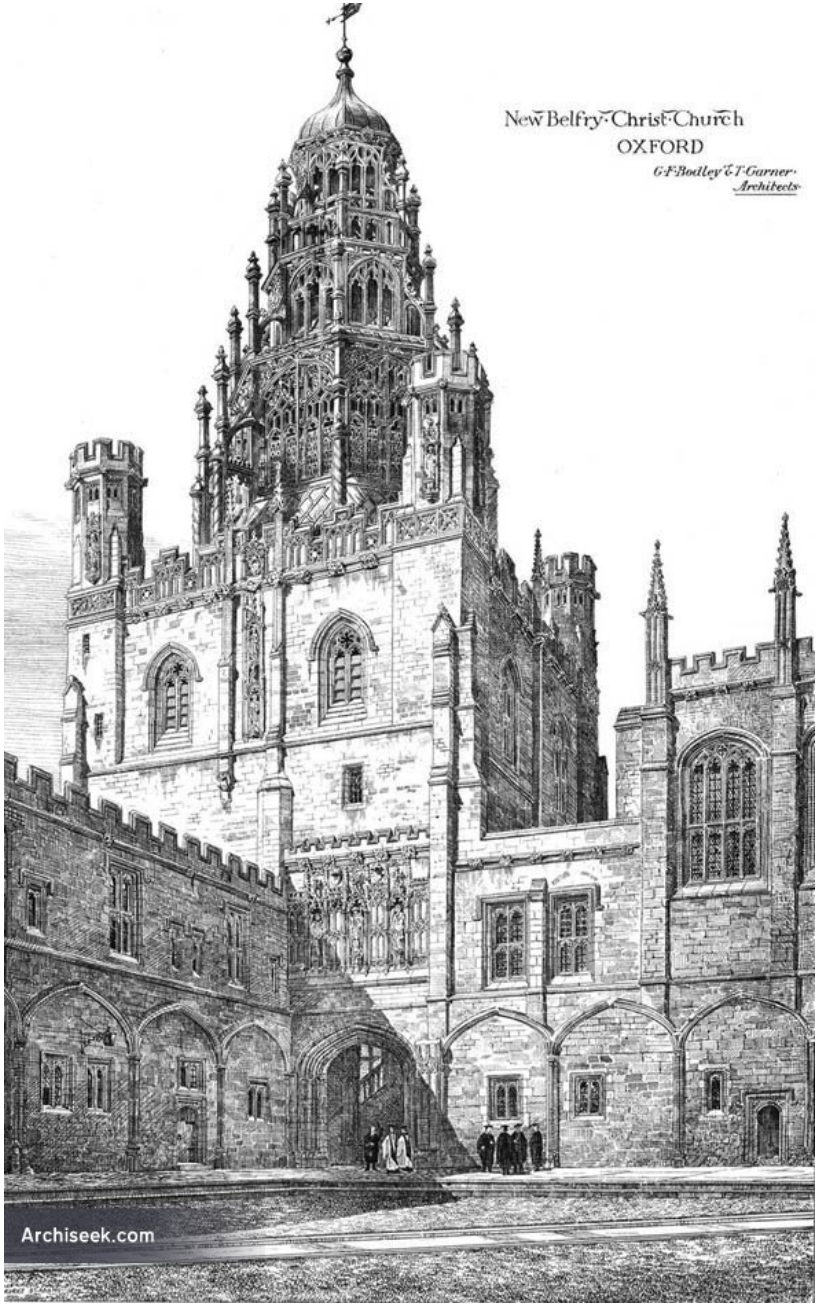
The winner of a Schulze election is the candidate X so that P(X,Y) > P(Y,X) for all possible Y. In this case it is B

i.e. it takes more voters to fancy X over Y than Y over X

Fair and fast. Widely used on the internet

Dodgson Method




New Belfry Christ Church
OXFORD
G.F.Bodley & T.Garner
Architects
Archiseek.com

Dodgson winner:

The candidate for whom the smallest number of (adjacent) changes are needed for them to be the Condorcet winner

Number of votes	Preferences
3	A > B > C > D
1	D > B > A > C
1	D > C > A > B
1	C > D > B > A
1	B > D > C > A

Only one of the first three voters has to change their mind for B to be the Condorcet winner

B is the Dodgson winner

Fair method, but VERY hard to use in practice (NP Hard)

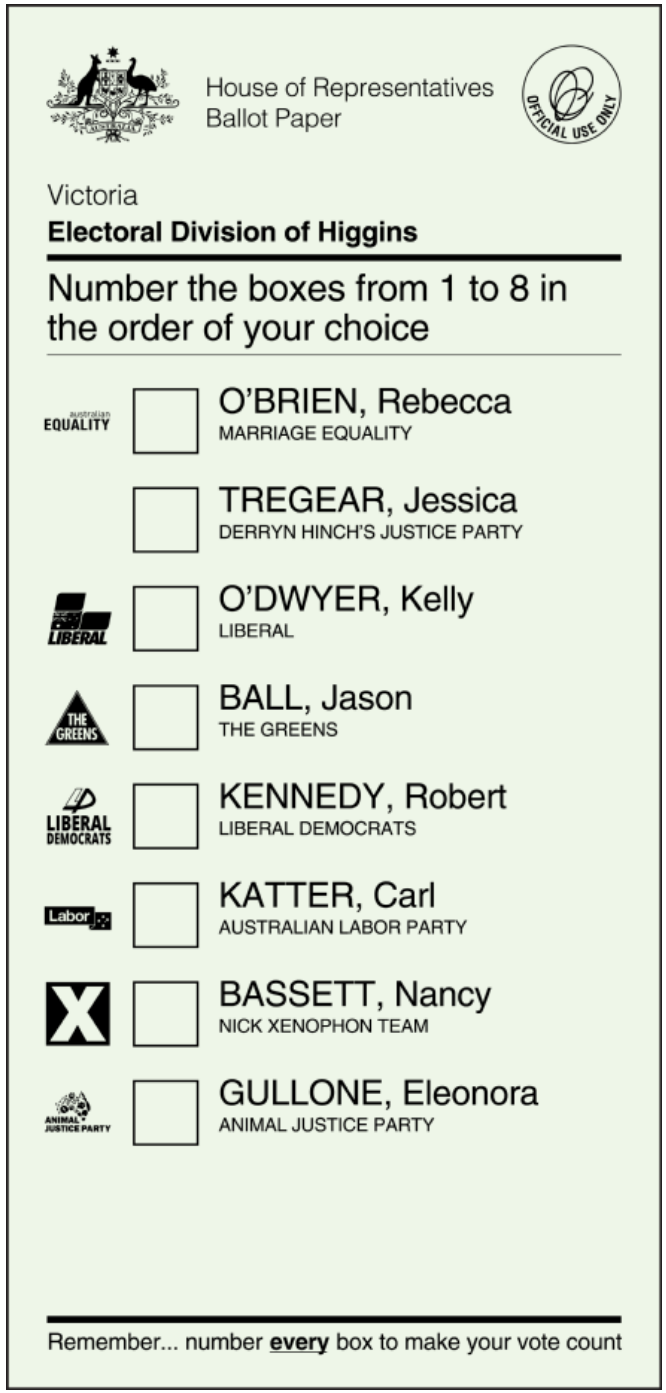
Reasonable practical compromises

IRV/AV, STV Voting Methods

Instant Run-Off Voting

Used in Australia to elect a single candidate

Voters express their preferences for each candidate


House of Representatives
Ballot Paper

OFFICIAL USE ONLY

Victoria
Electoral Division of Higgins

Number the boxes from 1 to 8 in the order of your choice

EQUALITY ☐ O'BRIEN, Rebecca
MARRIAGE EQUALITY

☐ TREGEAR, Jessica
DERRYN HINCH'S JUSTICE PARTY

LIBERAL ☐ O'DWYER, Kelly
LIBERAL

THE GREENS ☐ BALL, Jason
THE GREENS

LIBERAL DEMOCRATS ☐ KENNEDY, Robert
LIBERAL DEMOCRATS

Labor ☐ KATTER, Carl
AUSTRALIAN LABOR PARTY

X ☐ BASSETT, Nancy
NICK XENOPHON TEAM

ANIMAL JUSTICE PARTY ☐ GULLONE, Eleonora
ANIMAL JUSTICE PARTY

Remember... number **every** box to make your vote count

The ballots are initially counted for each voter's top choice If a candidate has more than half of the vote based on first-choices, that candidate wins

If not, the candidate with the fewest votes is eliminated

The voters who selected the defeated candidate as a first choice have their votes for that candidate added to the totals of their next choice

Process continues until a candidate has more than half of the votes

Candidates A,B,C Voters a,b,c,d,e

Round 1

	A	B	C
a	1	3	2
b	2	1	3
c	3	2	1
d	1	3	2
e	2	1	3
Total	2	2	1

B > A A > C C > B No Condorcet winner

C is eliminated on the first round

Round 2

	A	B
a	1	2
b	2	1
c	2	1
d	1	2
e	2	1
Total	2	3

B is the winner of the IRV vote

Note: A is the Borda and the Shulze winner

Single Transferable Vote (STV)

Invented in 1819 by Thomas Hill

Similar to the IRV method, used if there are multiple candidates elected to N posts

Used, for elections to the London Mathematical Society and many different countries

In each round the last candidate is removed and their votes reallocated. Process continues until only N candidates remain

STV is easy to use and approximately proportional, but does not necessarily deliver the Condorcet winner

Eurovision

2	DENMARK	281	SWEDEN	62
12	AZERBAIJAN	234	8	GEORGIA	50
10	UKRAINE	214	1	BELARUS	48
6	NORWAY	191	ICELAND	47
7	RUSSIA	174	ARMENIA	41
GREECE	152	UNITED KINGDOM	23
ITALY	126	3	ESTONIA	19
MALTA	120	GERMANY	18
4	THE NETHERLANDS	114	LITHUANIA	17
5	HUNGARY	84	FRANCE	14
MOLDOVA	71	FINLAND	13
BELGIUM	71	SPAIN	8
ROMANIA	65	IRELAND	5

LITHUANIA

1 2 3 4 5 6 7 8 10 12

39 of 39 countries voting

Eurovision Facts

Run every year since the 1950s

26 entrants in the final

Ridiculous staging, hilarious costumes,
cringe worthy announcers, sarcastic commentators

Waterloo from Abba in 1974.

1994 winner was the
interval music

1978 Norway Null Points

1968

Cliff Richard Congratulations

beaten by La La La

by one point!!

Widespread accusations of vote rigging

The voting at the end is by far the best part!

A true conflict between fair assessment of each song,
and outrageous tactical and political voting

Double Borda Method is used for the voting

Juries rank songs: Give 12,10,8,7,6,5,4,3,2,1,0 points

Countries rank songs by tele-voting: Give points as above

Jury votes accumulate. Tele-votes added at the end

Country	Jury score	Jury Rank	Tele-vote score	Tele-vote rank	Total score	Total rank
Sweden	365	1	272	3	627	1
Italy	171	6	356	1	527	2
Russia	234	3	286	2	520	3
Belgium	186	5	190	4	376	4
Australia	224	4	124	6	348	5
Latvia	249	2	88	8	337	6
Norway	163	7	37	10	200	7
Estonia	53	11	144	5	197	8
Israel	77	8	102	7	179	9
Georgia	62	10	51	9	113	10

Advantages of the method:

Real time scoring

Build up of tension as
the results are announced

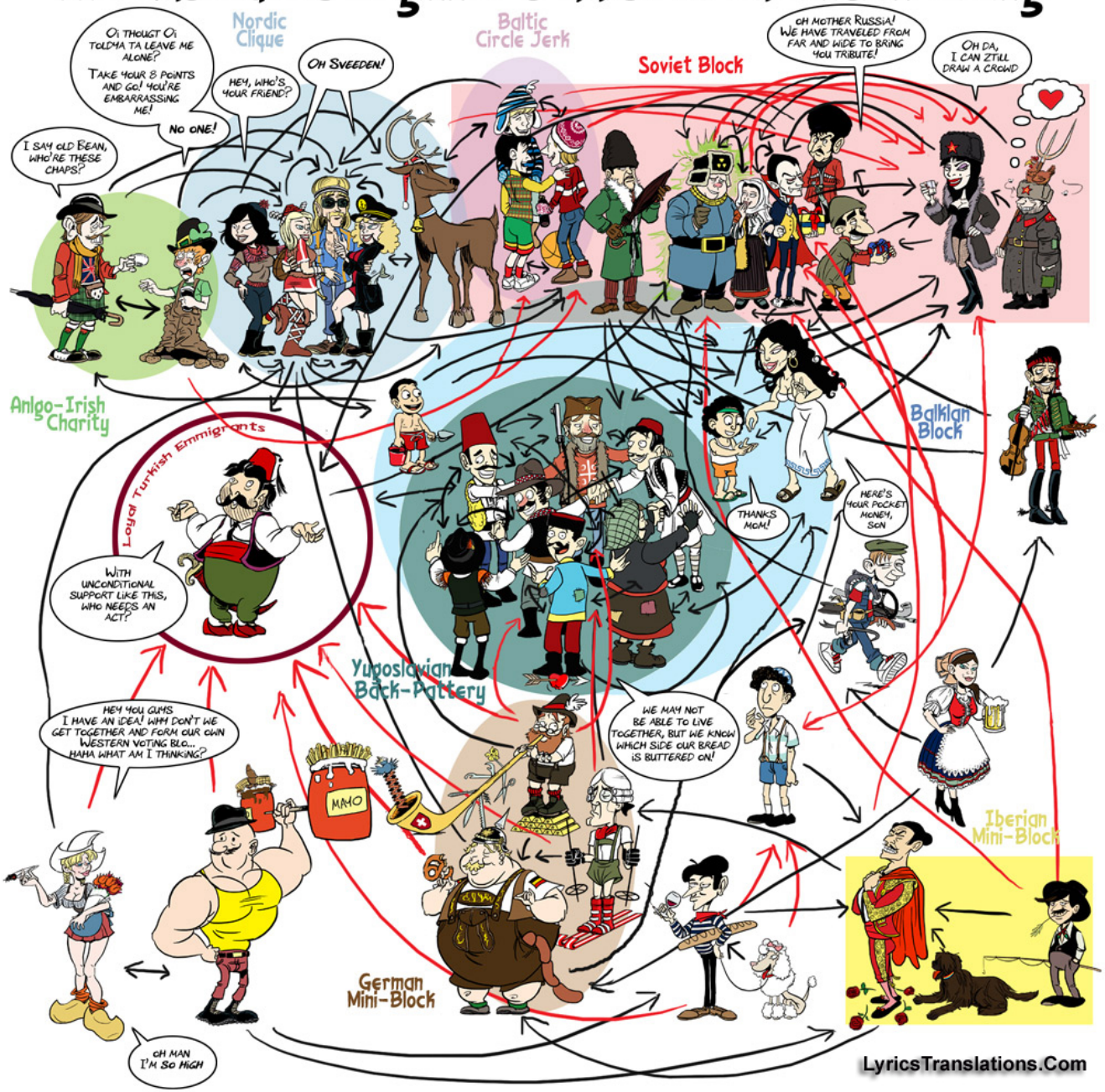
Disadvantages:

Not a Condorcet method,

Doesn't necessarily elect
the song favoured by the majority

Extremely vulnerable to tactical voting!!!!!

The Ilustrated Beginner's Guide to Eurovision Voting




POLLING
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Have fun on December 12th