Maths goes into Space

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Space is big business

$50 Billion investment

85 rocket launches in 2016

4 256 Satellites currently in orbit (that we know about)

Satellites are used in many technological applications

- Telecommunications (TV, Mobile phones)
- Remote sensing
- Weather forecasting
- GPS
- Spying
- Observing space
- Spotting whales and sheep

Telstar

Space technology has played a profound role in human civilization since the very beginning

Mathematics plays a vital role in making this technology possible

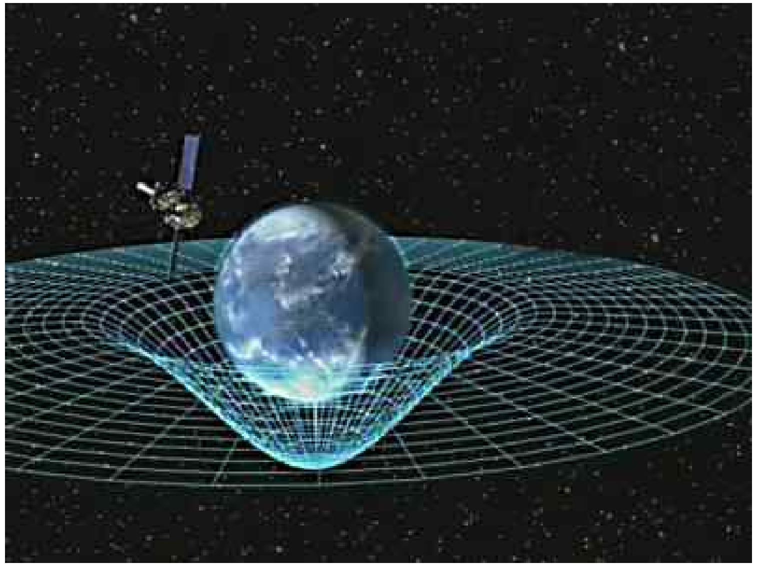
1. Space viewed from Earth

2. Earth viewed from Space

3. The Solar System

4. To Deep Space and back again

5. A bit of space origami



1. Space viewed from the Earth

Observing space from the Earth has profoundly changed human civilisation

And God said, Let there be lights in the firmament of the heaven to divide the day from the night; and let them be for signs, and for seasons, and for days, and years.

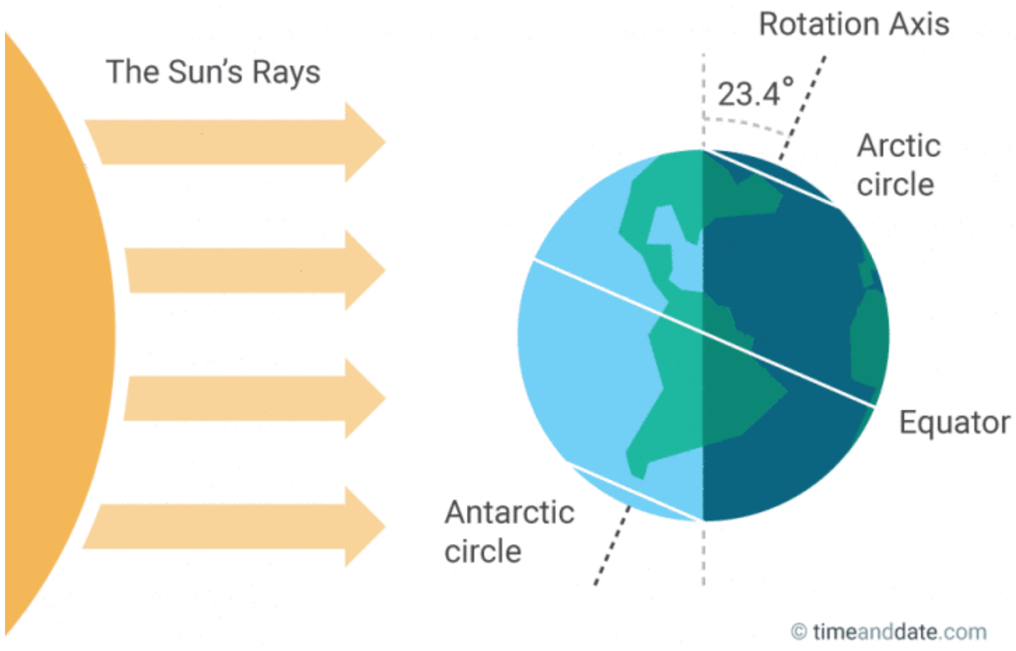
Earliest piece of mathematical technology: **The Calendar**

Need an accurate calendar to predict the seasons and know when to plant and harvest crops

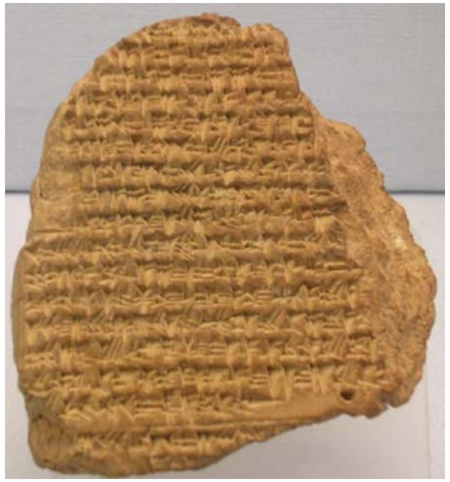
Need to be able to find the length of the year

Exact value: Year = 365.2422 days

Seasons determined by the tilt of the Earth's axis



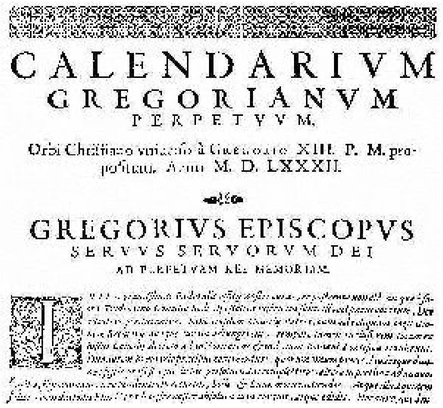
Babylonian calendar: 365 days



Egyptian and Julian calendars: 365.25 days

One leap year every four years. Used to predict Easter

Gregorian Calendar: 365.2425 days



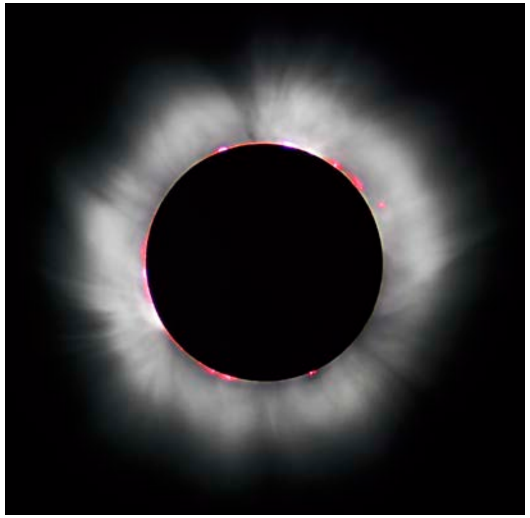
CALENDARIVM
GREGORIANVM
PERPETVVM.

Orbi Christiano vniuerso à Gregorio XIII. P. M. propositum. Anno M. D. LXXXII.

GREGORIVS EPISCOPVS
SERVVS SERVORVM DEI
AD PERPETVAM REI MEMORIAM.

(Wednesday) September 2, 1752, was followed immediately by (Thursday) September 14, 1752.

Calendar was also used to predict eclipses



These arise when the orbits of the sun and the moon coiincide and are predictable by looking for number patterns.

What numbers have remainder 2 when divided by 3, remainder 3 when divided by 5, and remainder 2 when divided by 7?

孫子算經卷上
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 勑注釋
度之所起起於忽欲知其忽蠶吐絲爲忽十忽
爲一絲十絲爲一毫十毫爲一氂十氂爲一分
十分爲一寸十寸爲一尺十尺爲一丈十丈爲
一引五十尺爲一端四十尺爲一疋六尺爲一
步二百四十步爲一畝三百步爲一里
稱之所起起於黍十黍爲一絫十絫爲一銖二
十四銖爲一兩十六兩爲一斤三十斤爲一鈞

Chinese Remainder Theorem

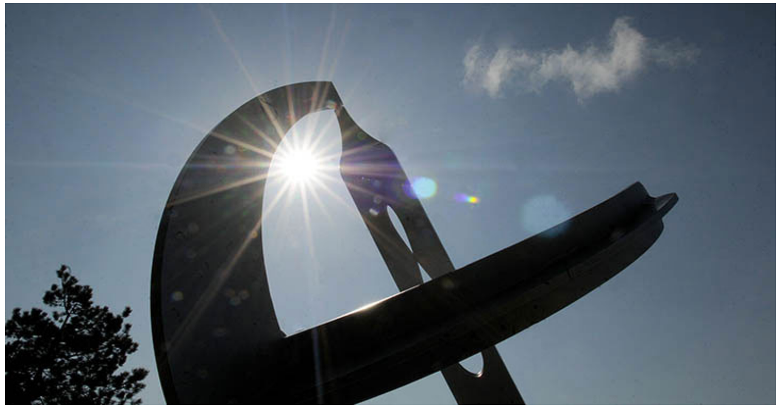
Described by C. F Gauss

Now used in Cryptography.

Telling the time

Traditional sundial

Corrected sundial



Analemmatic sundial

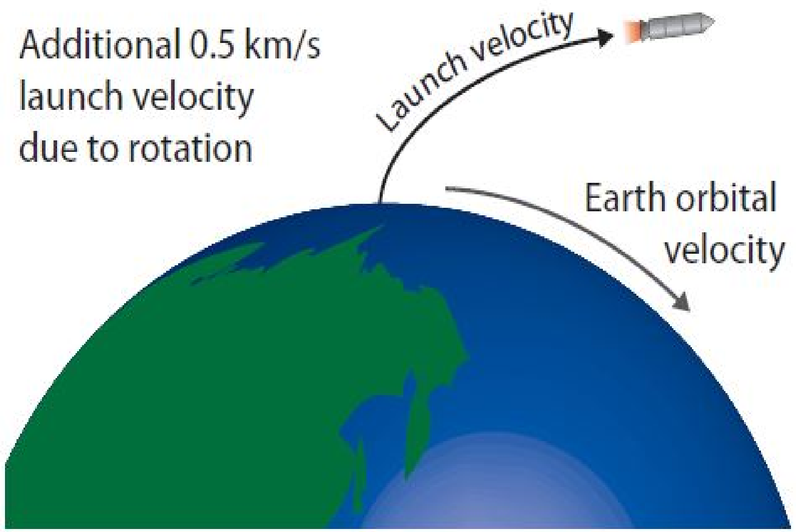
2. Earth viewed from Space

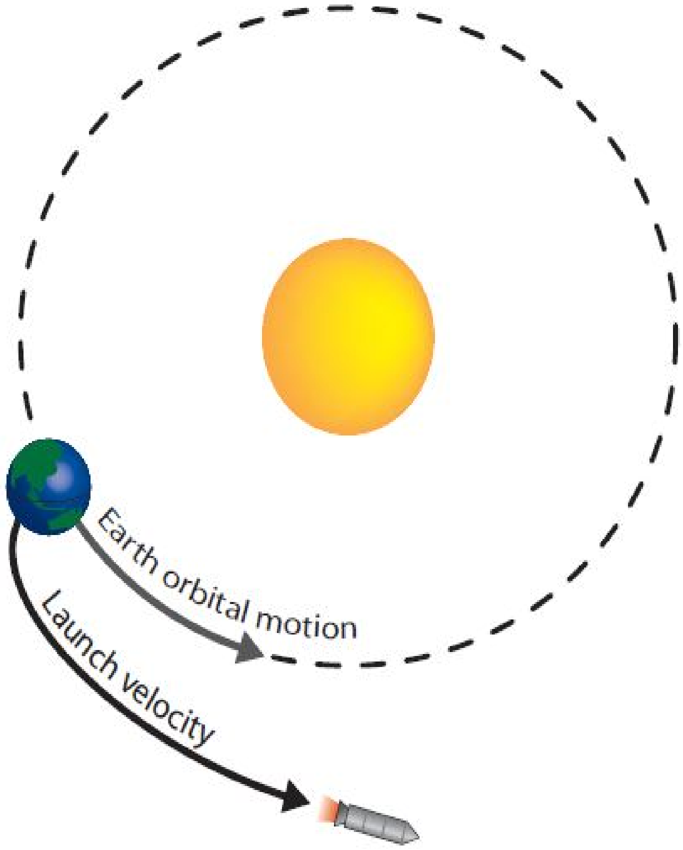
Oberth, Werner von Braun, Sergi Korolev

Pioneers of launching satellites into space using multi stage rockets

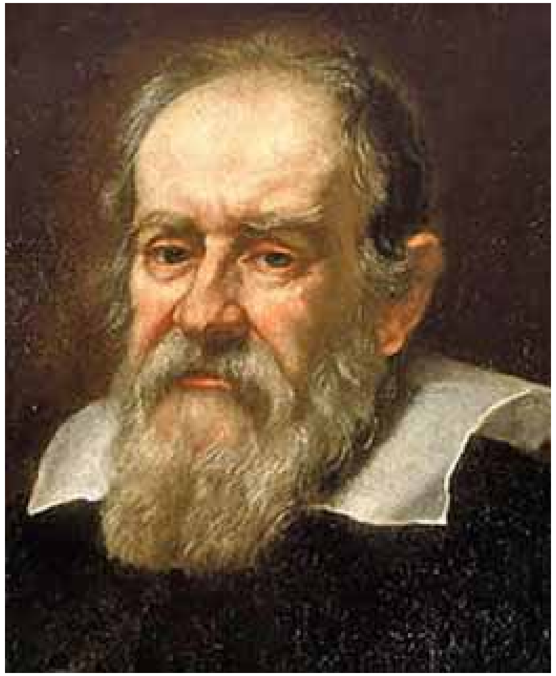
Sputnik 1957

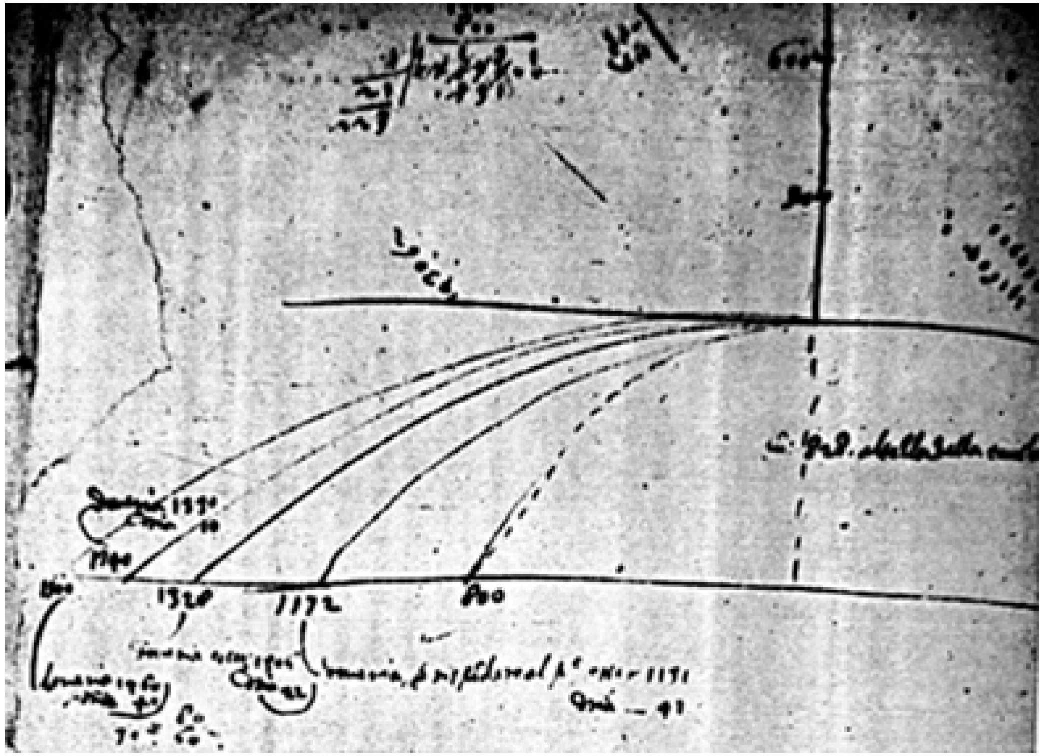
Rockets are launched vertically then turn to insert satellite into Earth orbit



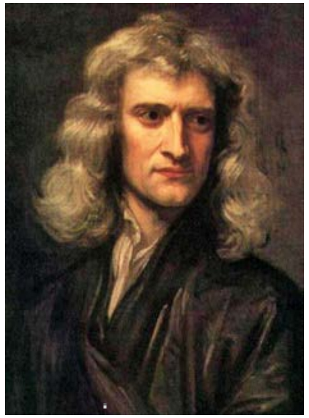


Basic mathematics worked out by Galileo and Newton

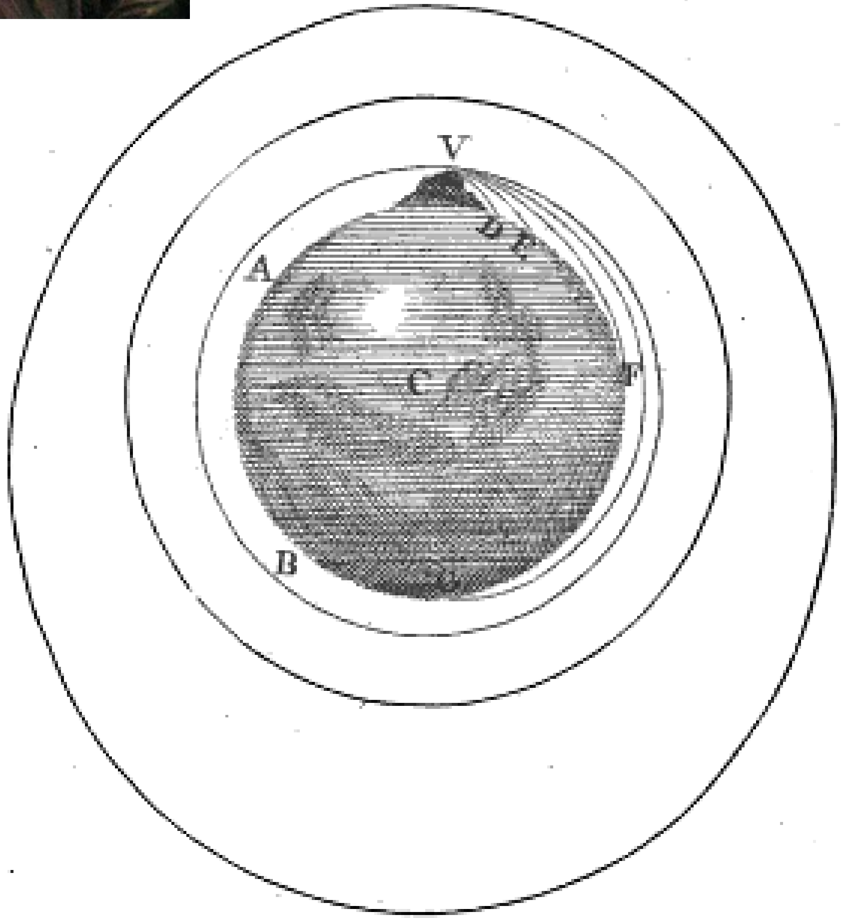




1600s Basic laws of ballistics



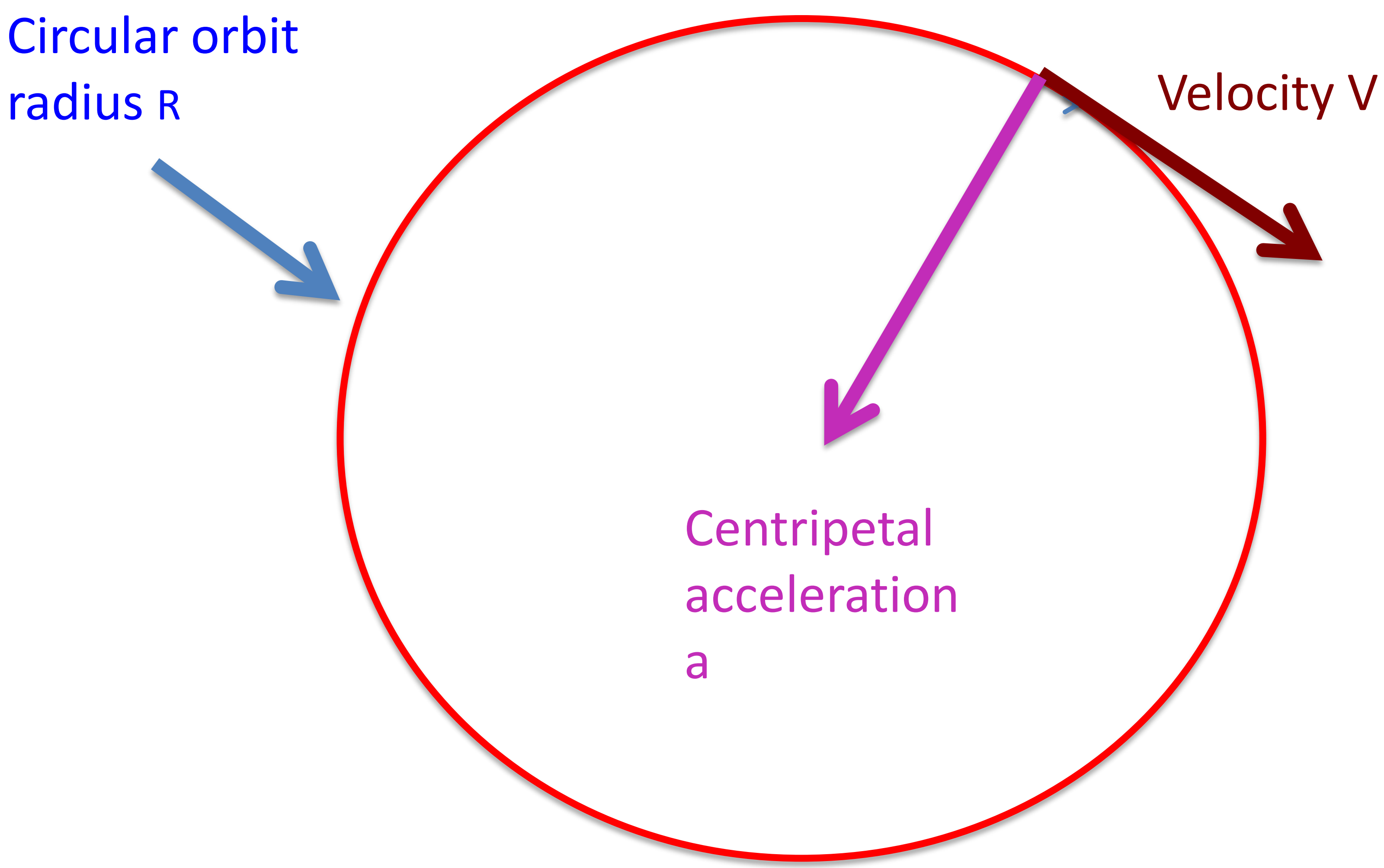
1690s Cannon Ball thought experiment



THAT by means of centripetal forces, the Planets may be retained in certain orbits, we may eaſily underſtand, if we conſider the motions of projectiles. For a ſtone projected is by the preſſure of its own weight forced out of the rectilinear path, which by the projection alone it ſhould have purſued, and made to deſcribe a curve line in the air; and through that crooked way is at laſt brought down to the ground. And the greater the velocity is with which it is projected, the farther it goes before it falls to the Earth. We may therefore ſuppoſe the velocity to be ſo encreaſed, that it would deſcribe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at laſt exceeding the limits of the Earth, it ſhould paſs quite by without touching it.

The effects of centripetal forces.

Centripetal acceleration



Circular orbit radius R

$$a = \frac{V^2}{R}$$

$$R = 6371km,$$
$$G = 6.67 \times 10^{-11}$$
$$M = 5.97 \times 10^{24} kg$$

Acceleration due to gravity at the Earth's surface

$$g = \frac{GM}{R^2} = 9.8ms^{-2}$$

$$V = 7.9\ km\ s^{-1}$$

Satellite at a different height

$$V = \sqrt{\frac{GM}{R}}$$

Orbital period

$$T = \frac{2\pi R}{V}$$

If T = 24 Hours

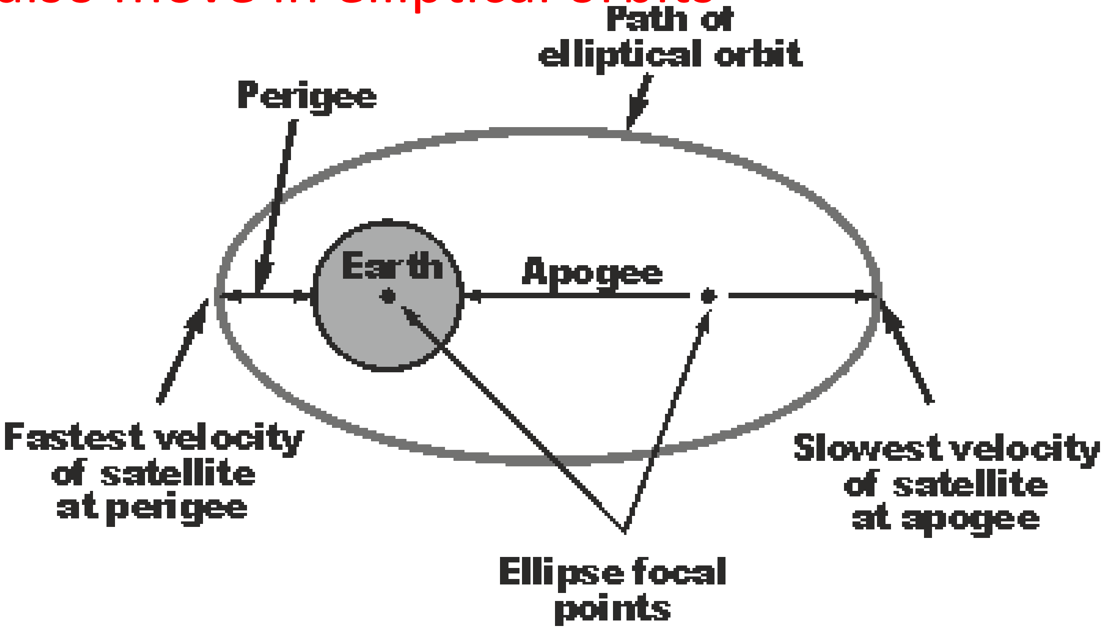
R = 42 200 km

Satellite is *Geostationary*

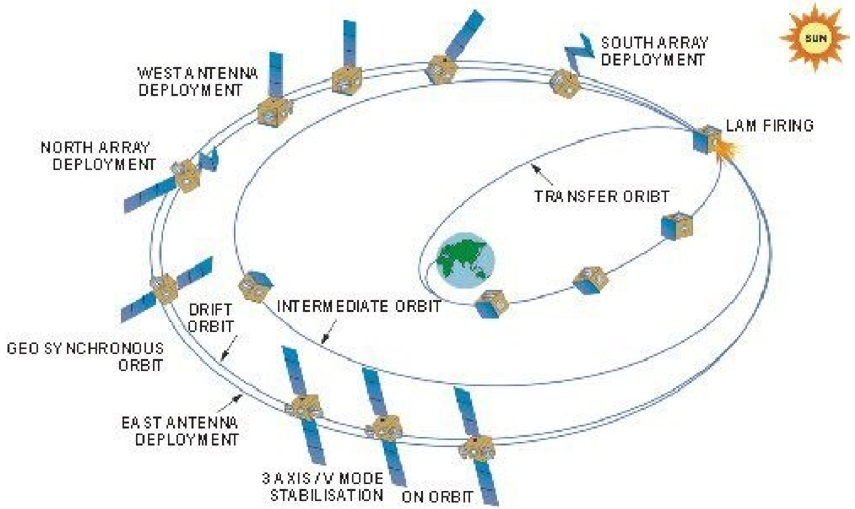
Very important for telecommunications and has led to a revolution in communications and social media

Arthur C Clarke paper in 1945

Satellites also move in elliptical orbits



And move between orbits



Satellites are tracked, guided, navigated and controlled as they move from one orbit to another

Typically moved by making small burns of fuel to change their velocity

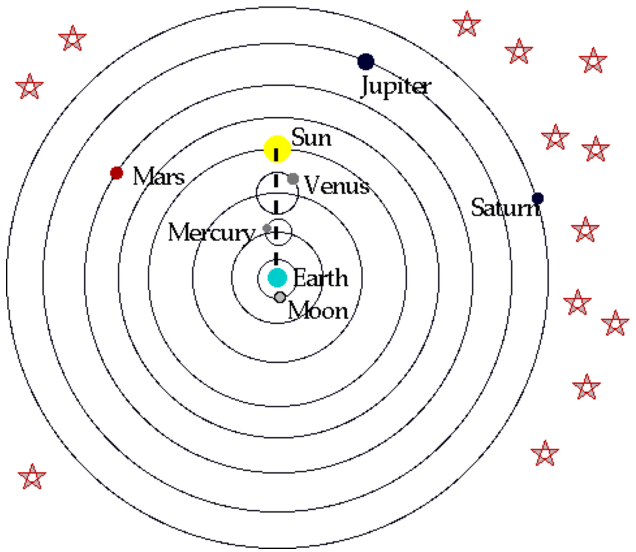
$$m_{fuel} = m_{initial}\left(1 - e^{-\frac{\Delta V}{Isp}}\right).$$

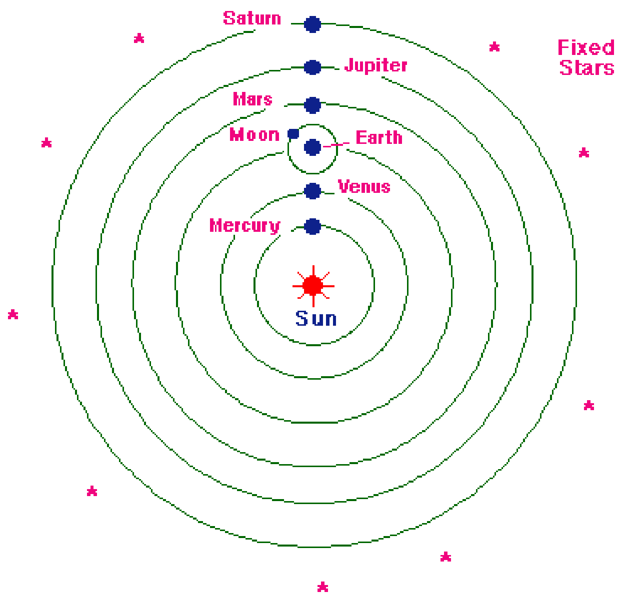
Optimise the orbit so that the change in velocity due to burning fuels is as small as possible. Requires very careful mathematical calculations.

3. The Solar System

Many theories for the motion of the planets in the solar system

Ptolemaic: Planets orbit the Earth on circles on circles





Copernican: Planets orbit the Sun on perfect circles

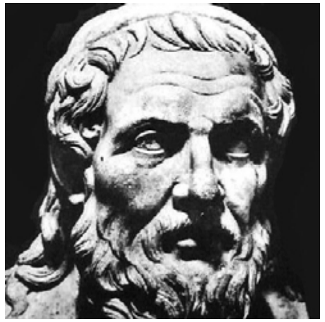
Neither quite worked

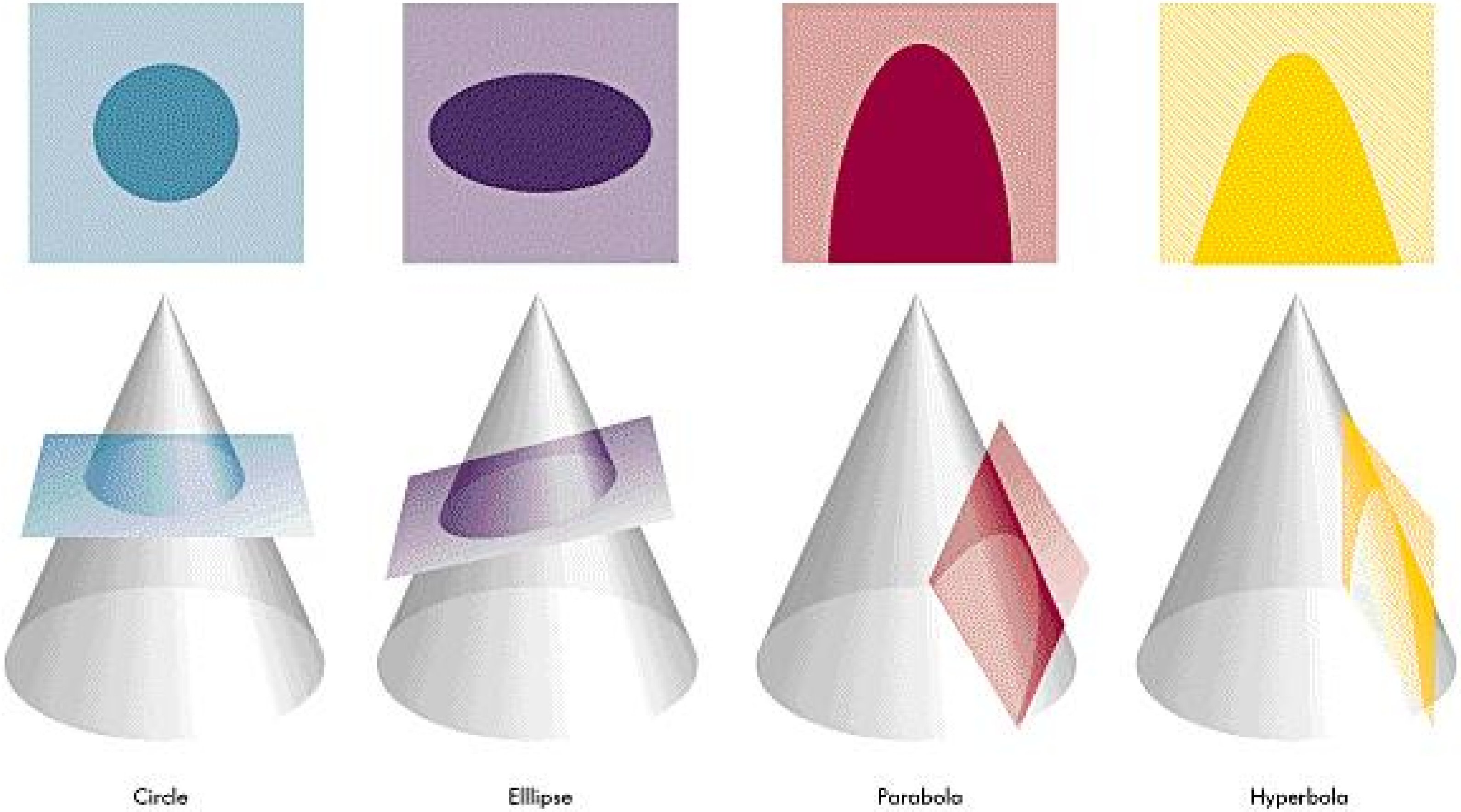
Copernican model caused huge controversy

Tycho Brahe: Collected data. Wanted to disprove Copernicus

Kepler 1600s analysed the data

Apollonius of Perga and conic sections





Equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

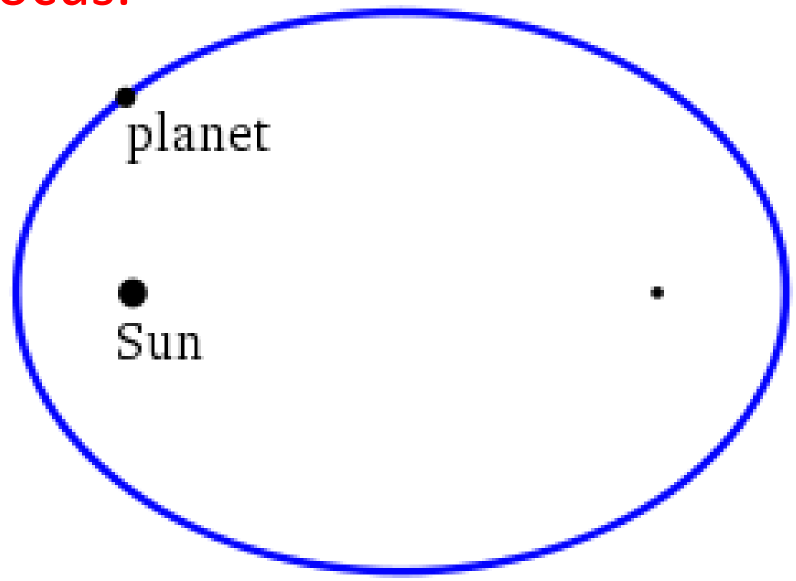
Polar equation

e < 1 Ellipse

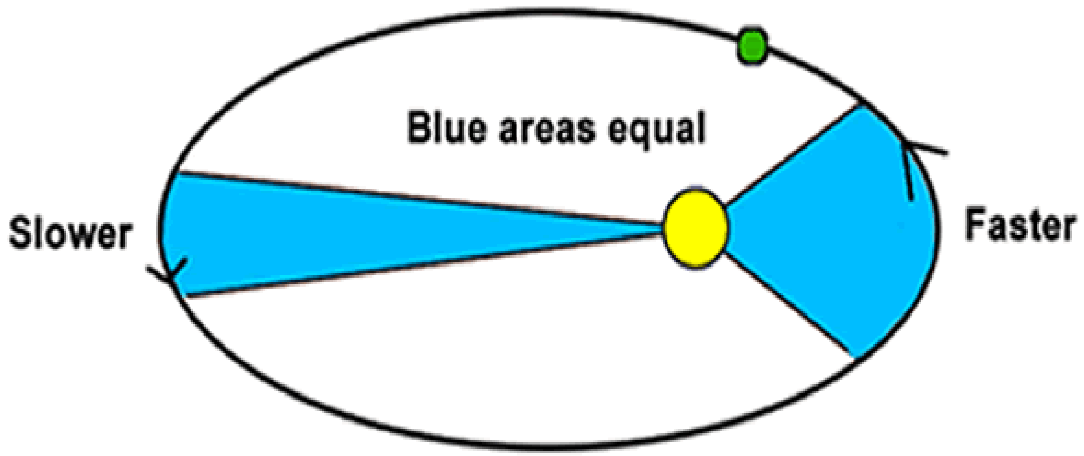
e > 1 Hyperbola

$$r = \frac{l}{1 + e\cos(\theta)}$$

Keplers first law: Planets move round the Sun in an ellipse with the Sun at its focus.

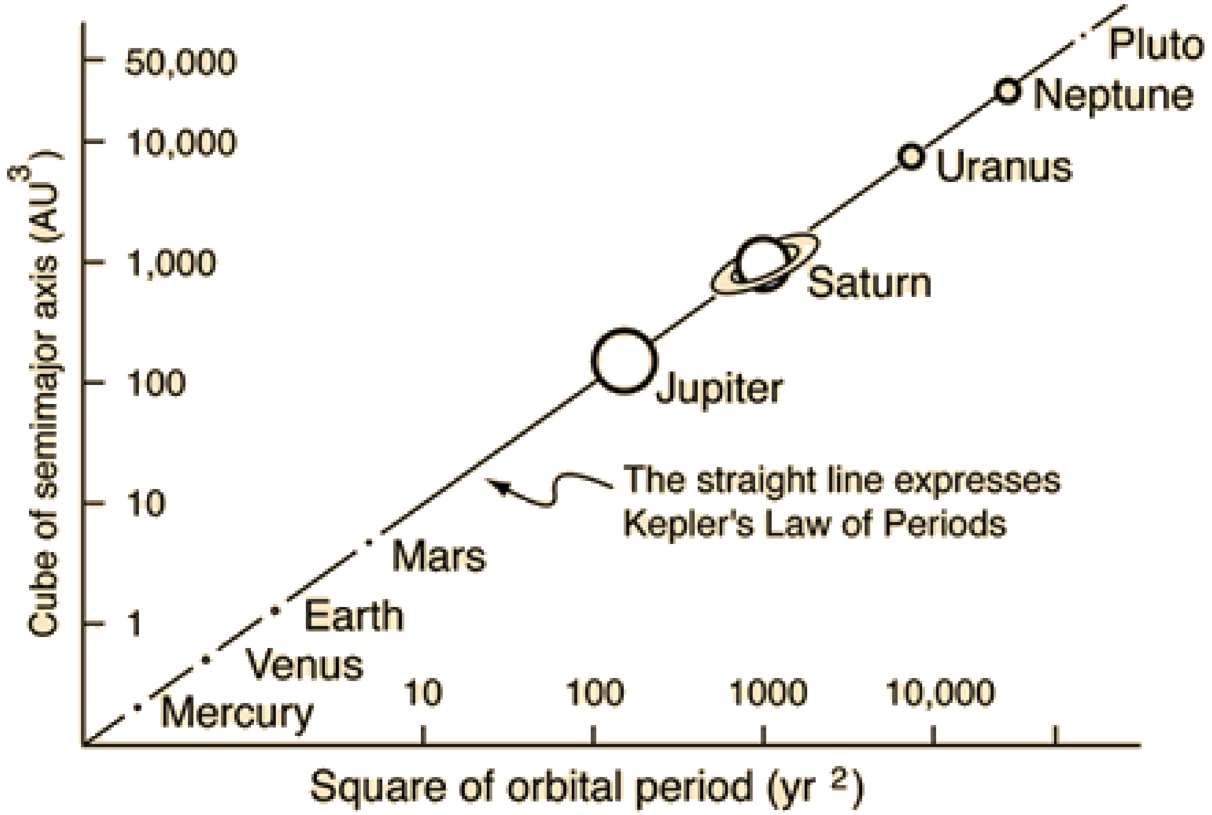


First Lucky fluke!!!

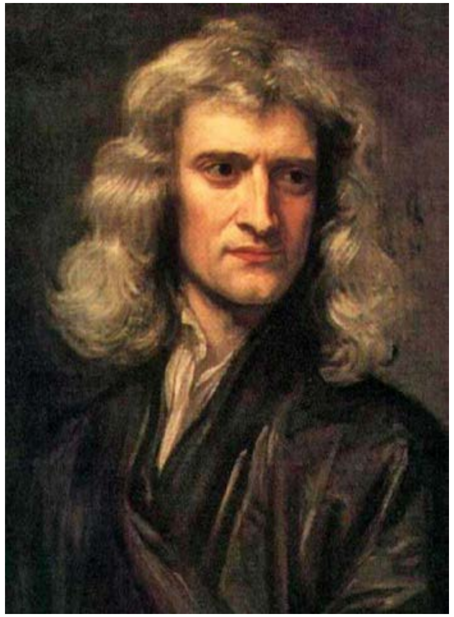


Second law. Equal areas are swept out in equal time .. Conservation of angular momentum

Third law: Orbital period squared was proportional to the planetary distance squared

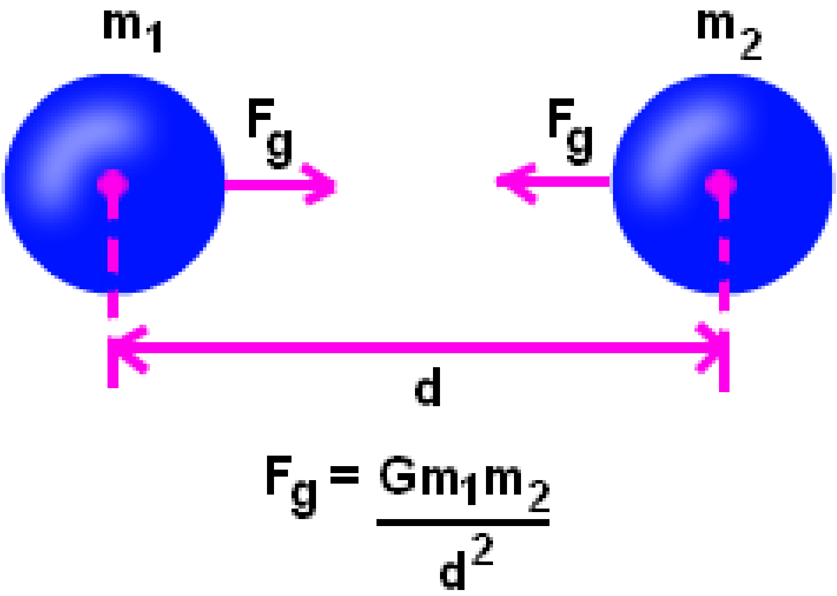


Showed HUGE MATHEMATICAL ORDER to the universe



1690 Newton explained this order through his laws of mechanics

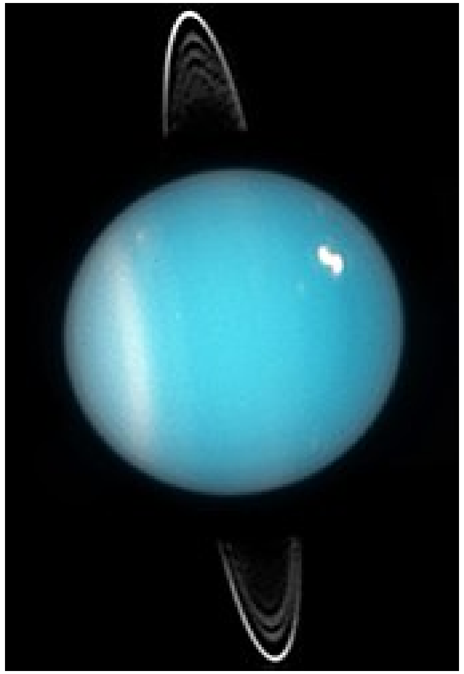
Gravitational force between two bodies was inversely proportional to the square of the distance between them



Second lucky fluke!

Could solve the system of the Sun and a planet **exactly.** Predicted an exactly elliptical orbit!

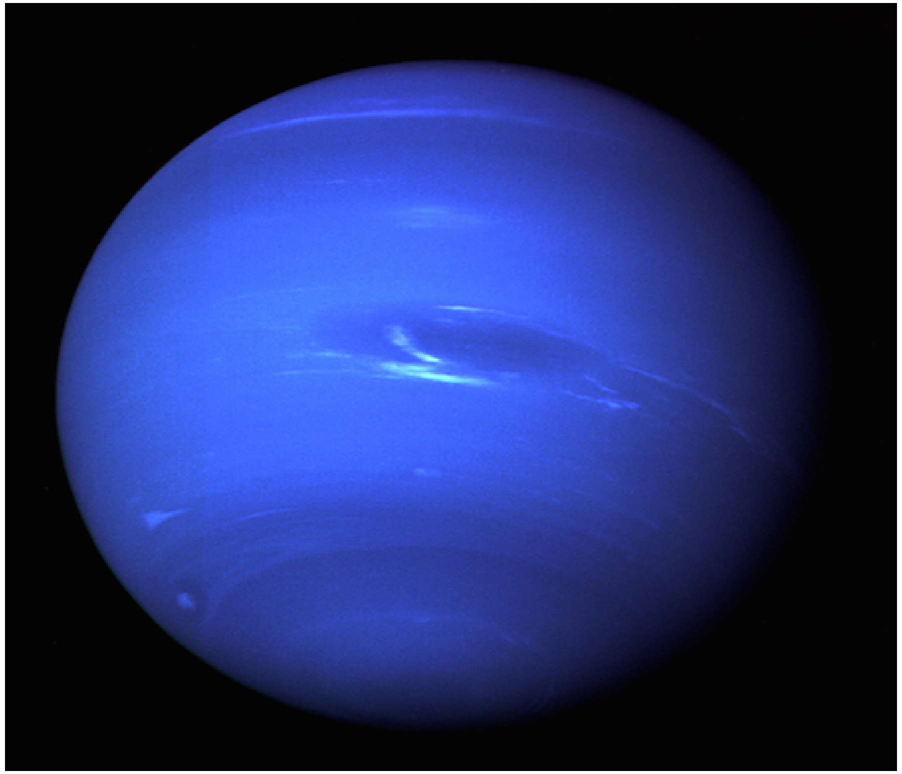
Led to great confidence in Newton's laws which worked brilliantly for all of the known planets



Uranus. Discovered in Bath by William Herschel on 13th March 1781

Orbit didn't quite agree with the predictions of Newton's laws

Neptune

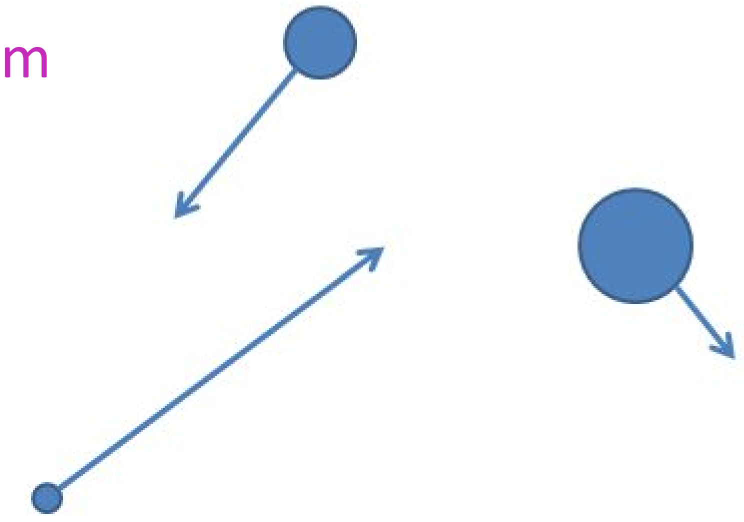


Existence and location predicted by Adams and Le Verrier

Discovered by Galle following these predictions

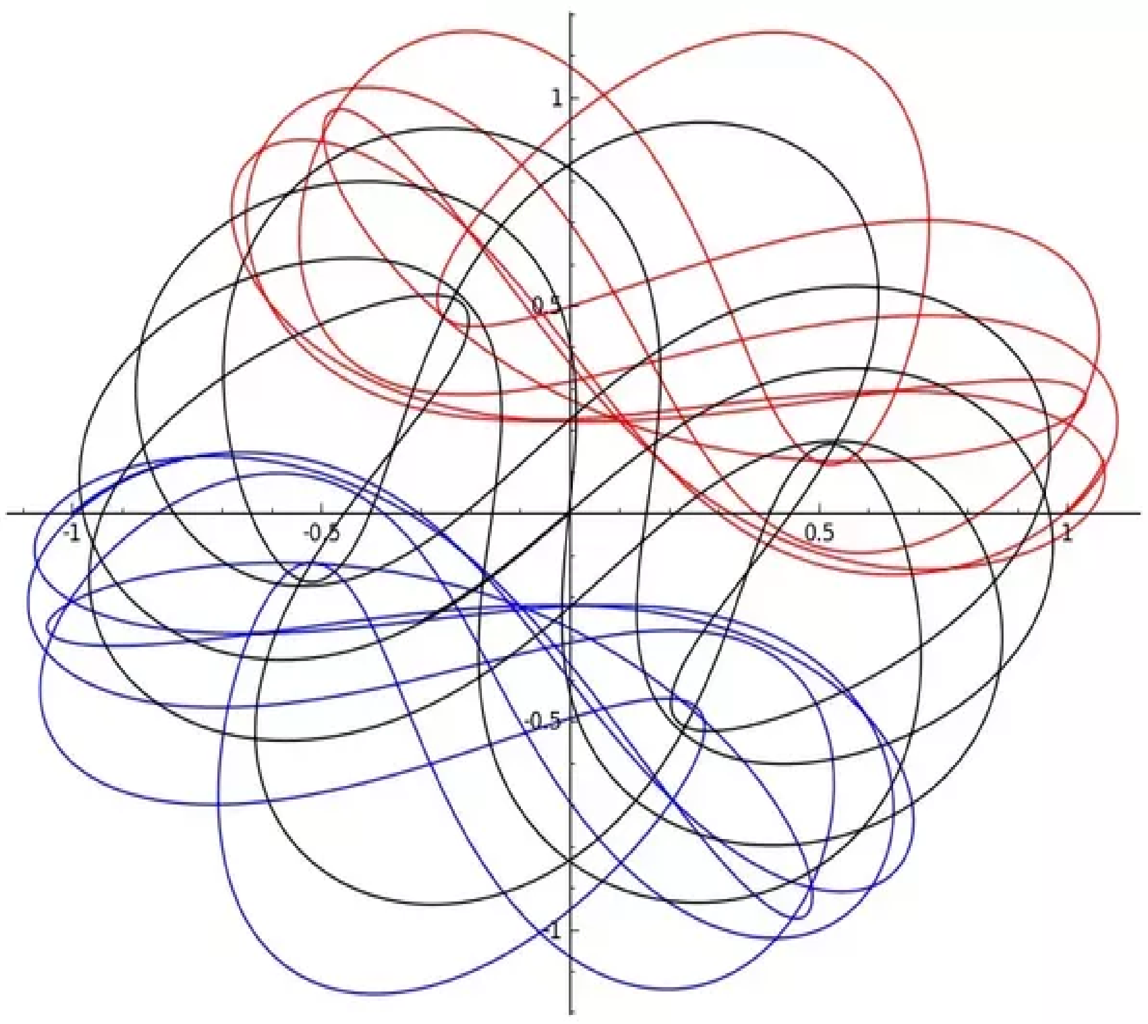
A mathematical model leads to the discovery of something quite new!!

The Three Body Problem



$$\frac{d^2\, \mathbf{r}_i}{dt^2} = -G \sum_{j \neq i}^{N} \frac{m_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Problem: No simple solution to the three body problem!!!!



Typical chaotic orbit in the three body problem

Studied by the mathematician Poincare who was looking at the stability of the Solar system.

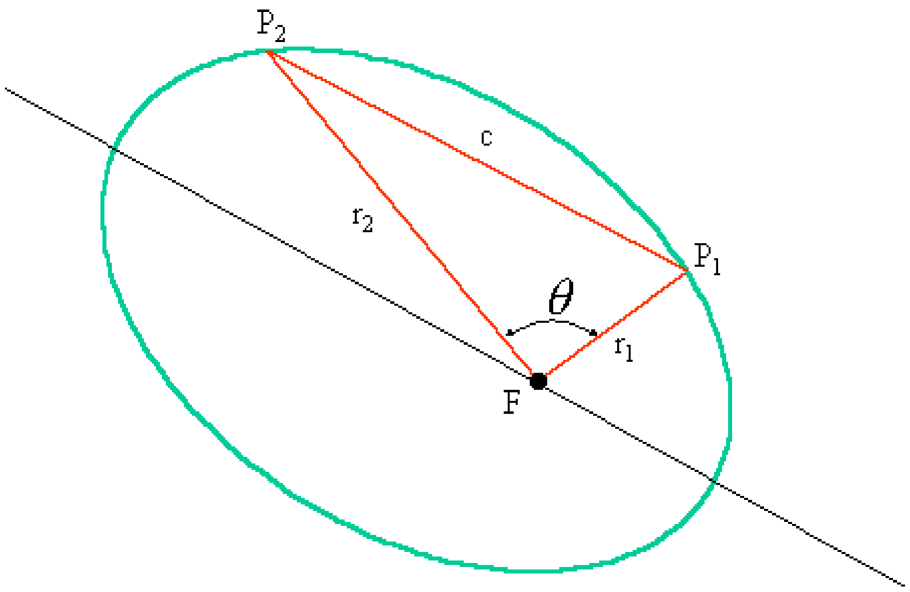
Also important in working out whether we will be hit by an asteroid!

Lambert’s Problem

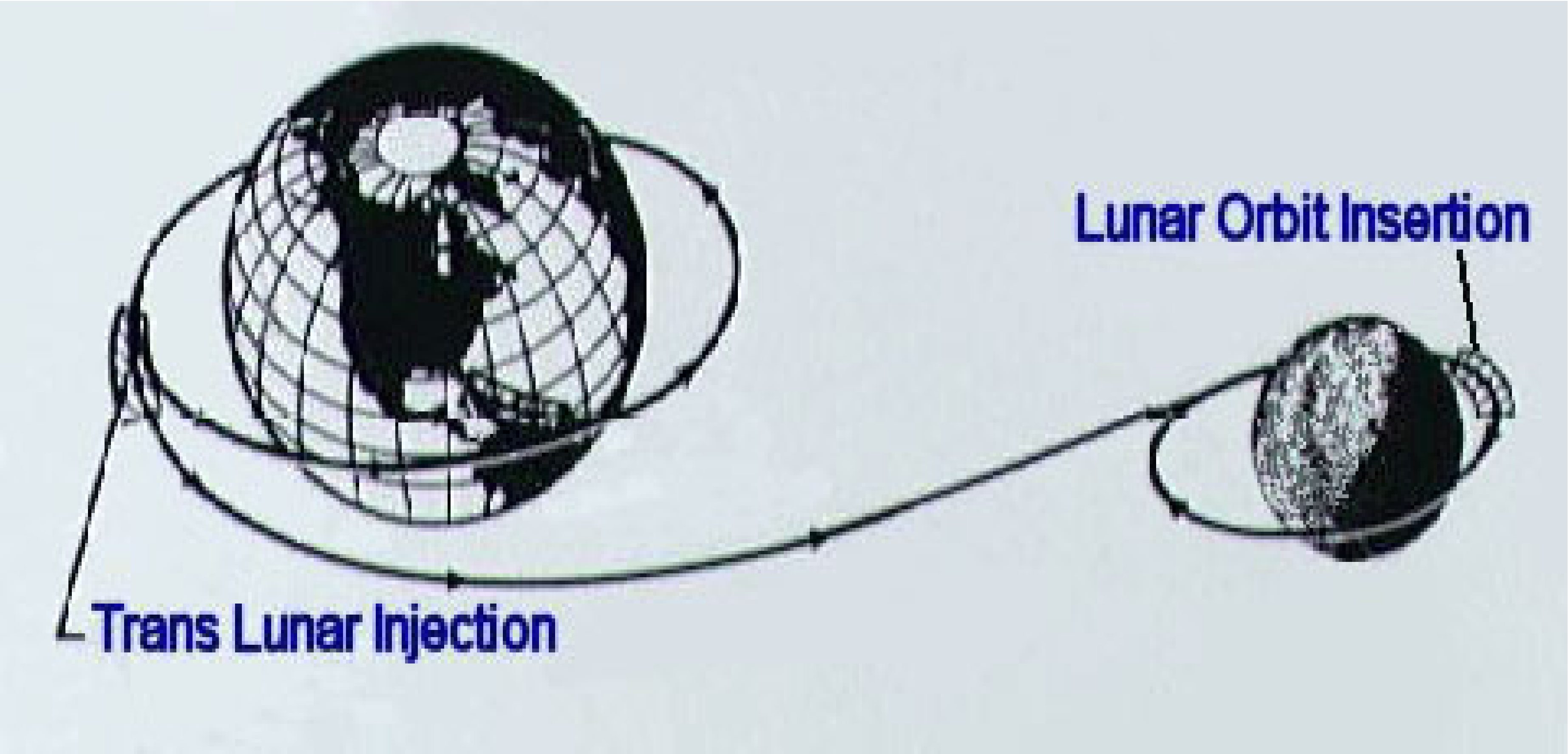
But .. If one of the three bodies is small (such as a satellite) then we can solve it.

Solved by Swiss mathematician Lambert 1728-1777

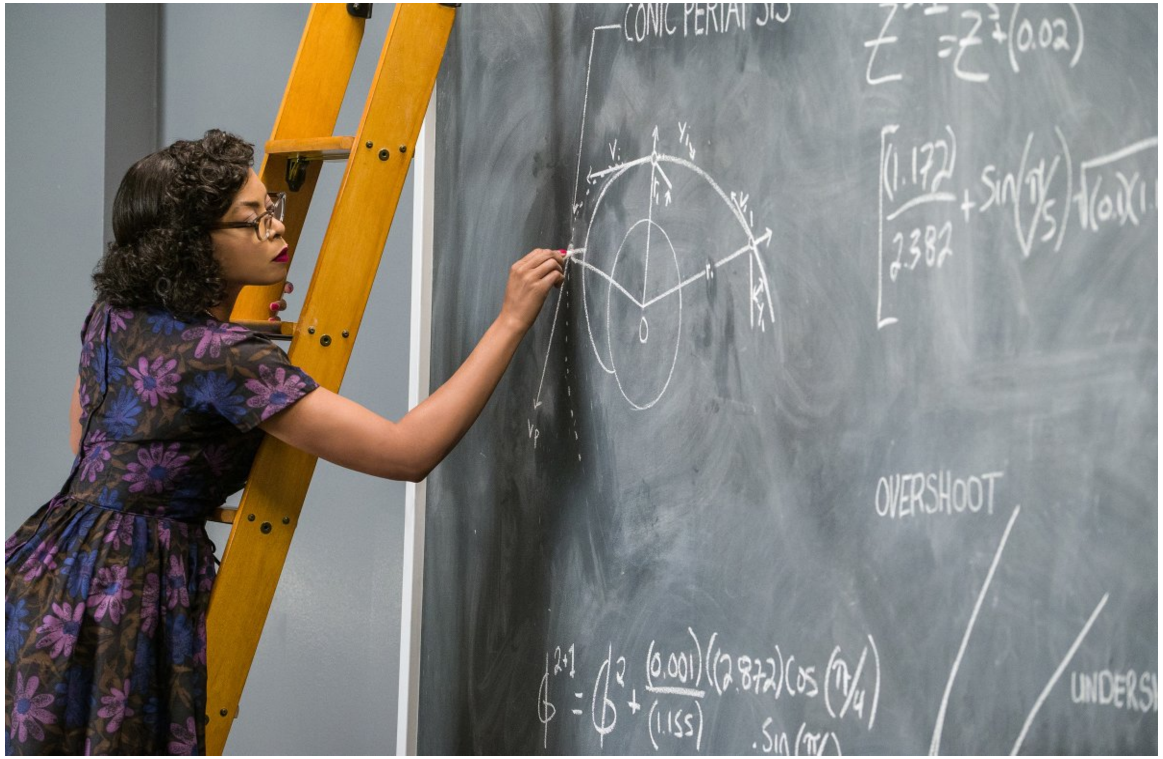
Do this to work out the orbit of a satellite around the sun and the planets



Solved to put the Apollo space craft on the moon and bring Apollo 13 back again



Three of the orbit calculators were African American women including the mathematician Katherine Goble the engineer Mary Jackson and their supervisor Dorothy Vaughan



Slingshots and hyperbolic orbits

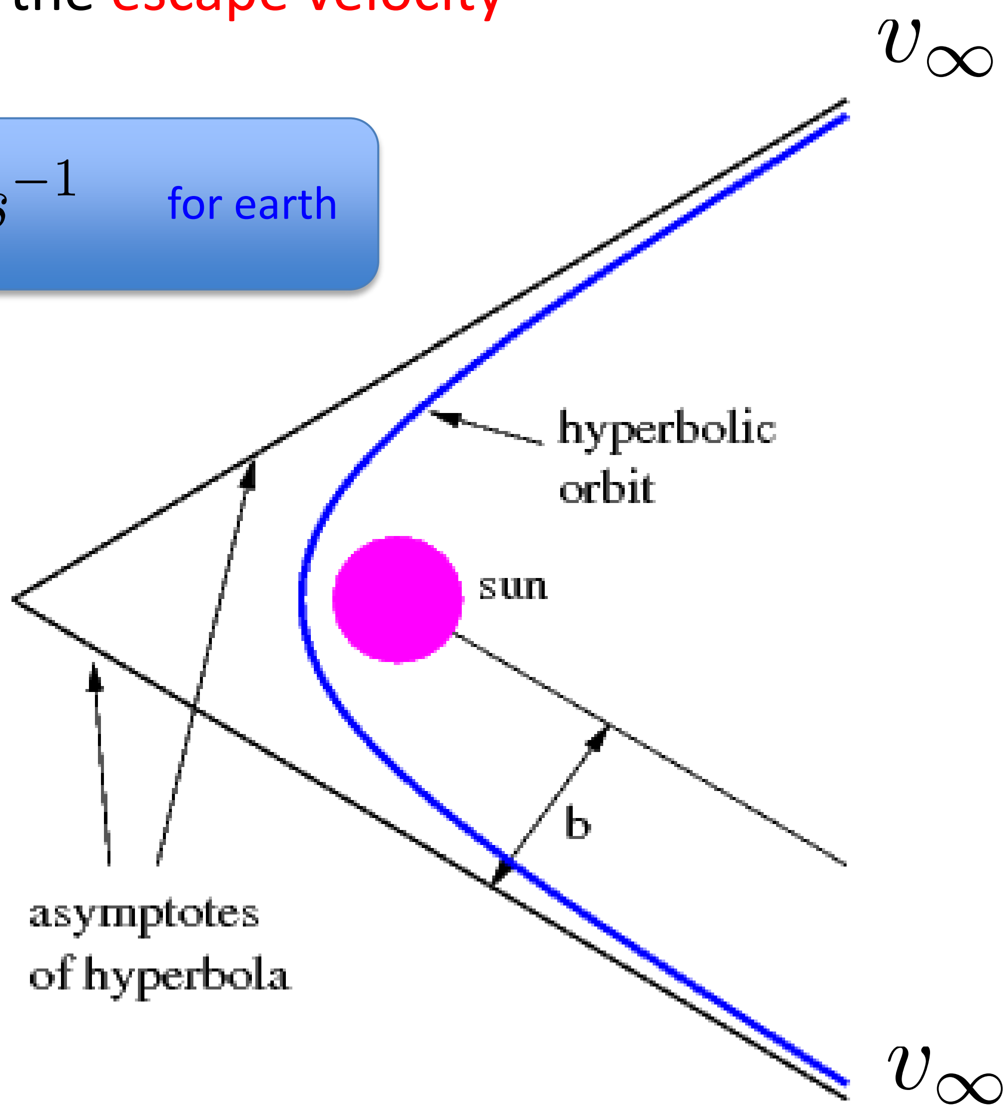
If the velocity is greater than the escape velocity

$$V_e = \sqrt{2gR} = 11.2\ km\ s^{-1}$$ for earth

Then the satellite moves on a hyperbolic orbit

Arrives and leaves with relative velocity

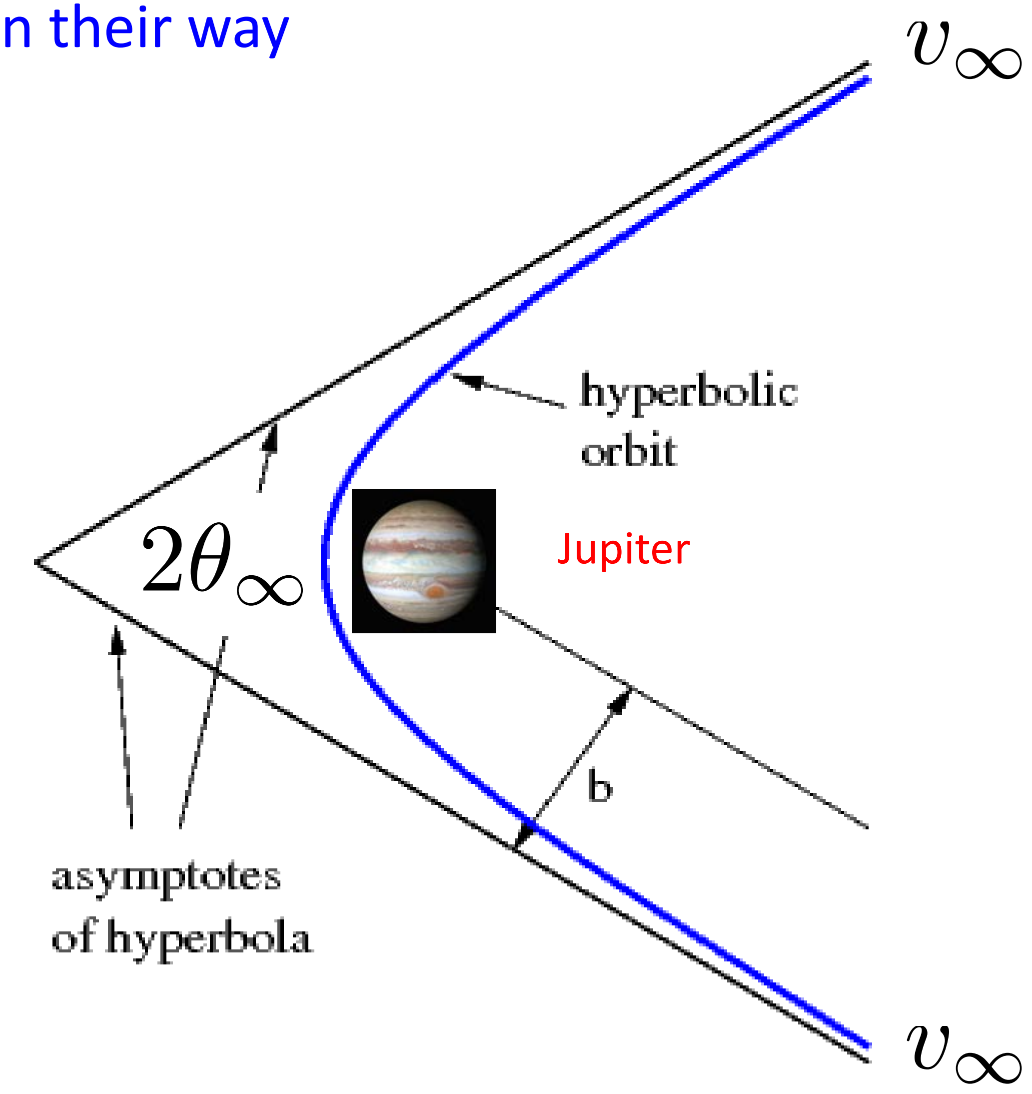
$$v_\infty^2 = v^2 - V_e^2$$



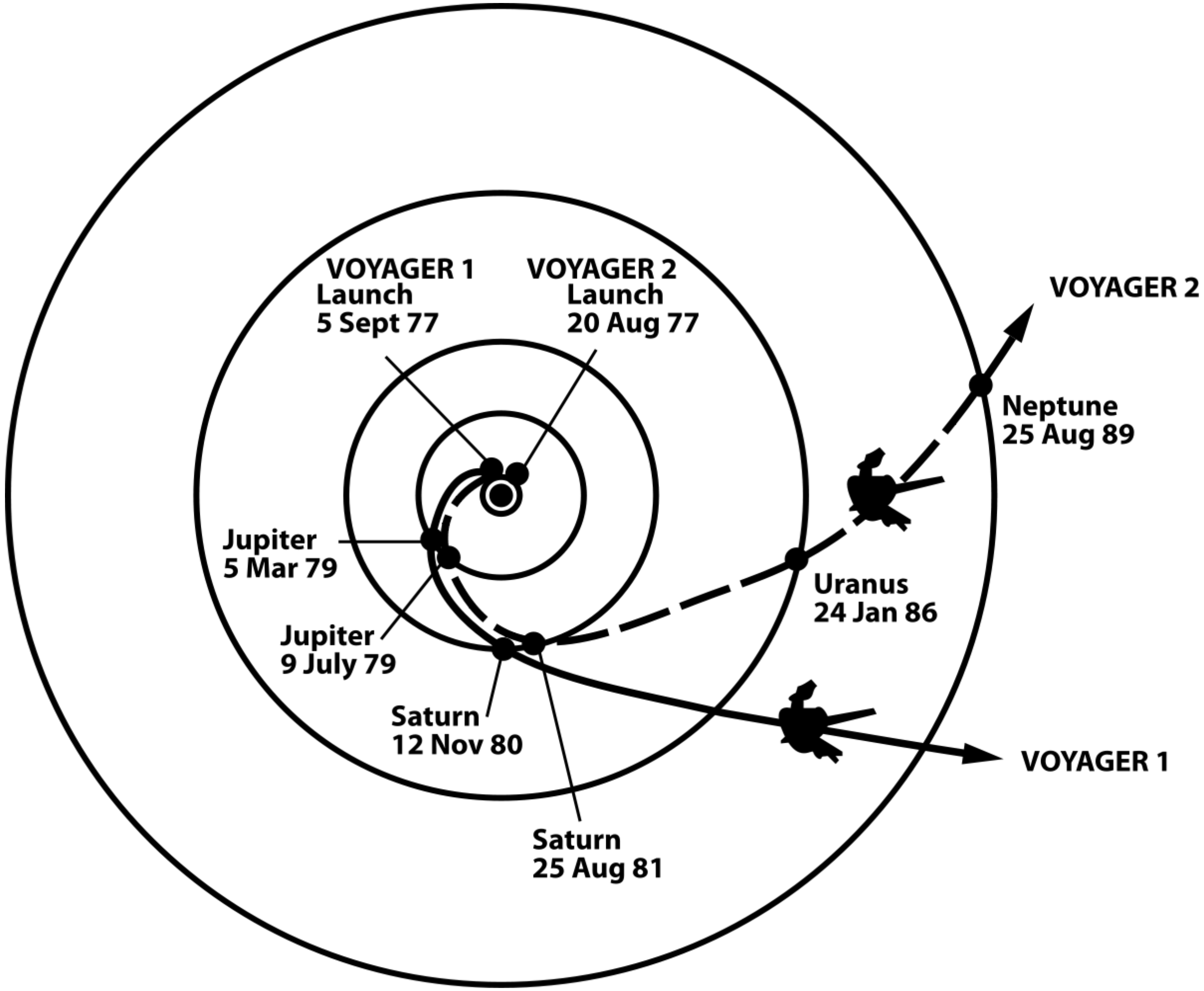
This is exploited in the slingshot effect to accelerate satellites in their orbits by using a gravity assist from planets on their way

$$\tan(\theta_\infty) = \frac{b\, v_\infty^2}{\mu}$$

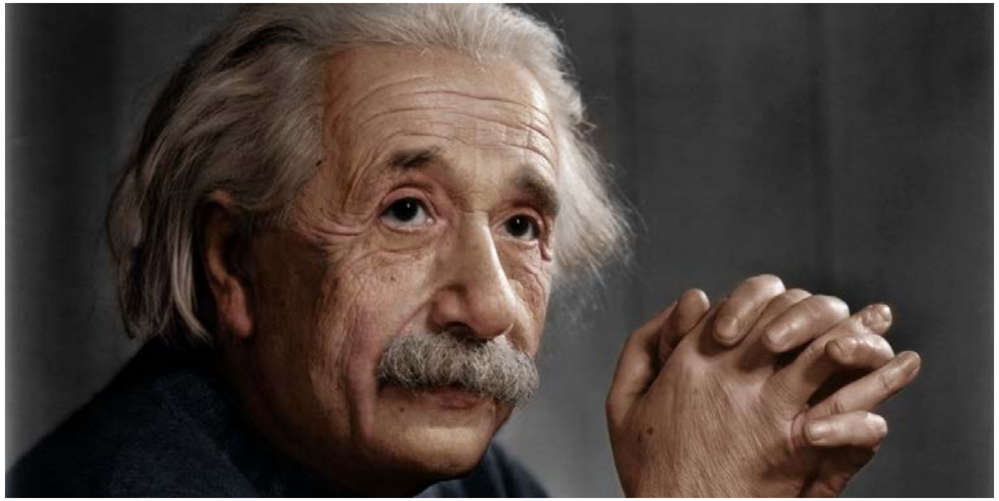
$$\mu = GM$$



Gravity assists for the Voyager probes

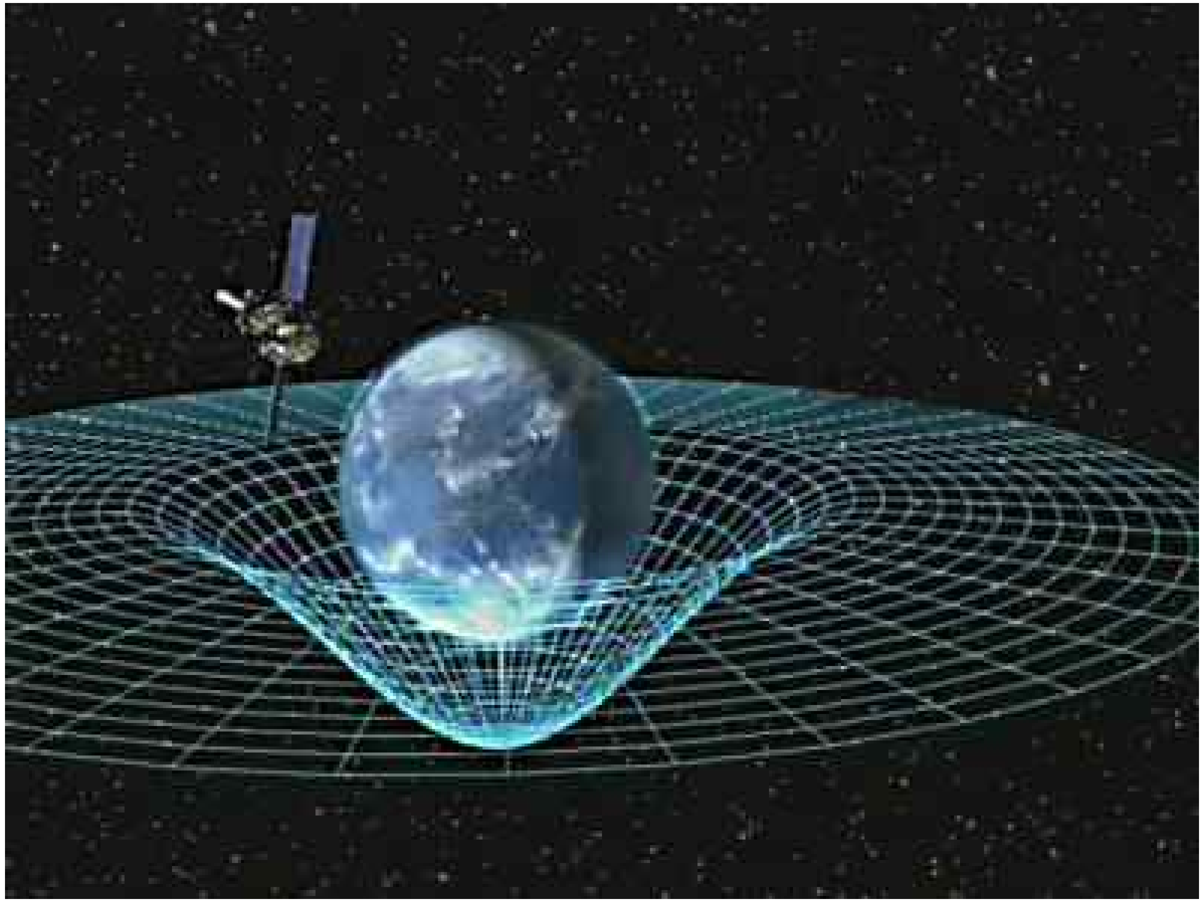


4. To Deep Space and Back Again



Einstein's General Theory of Relativity 1915

Gravitational acceleration is due to the distortion of space-time by a massive body



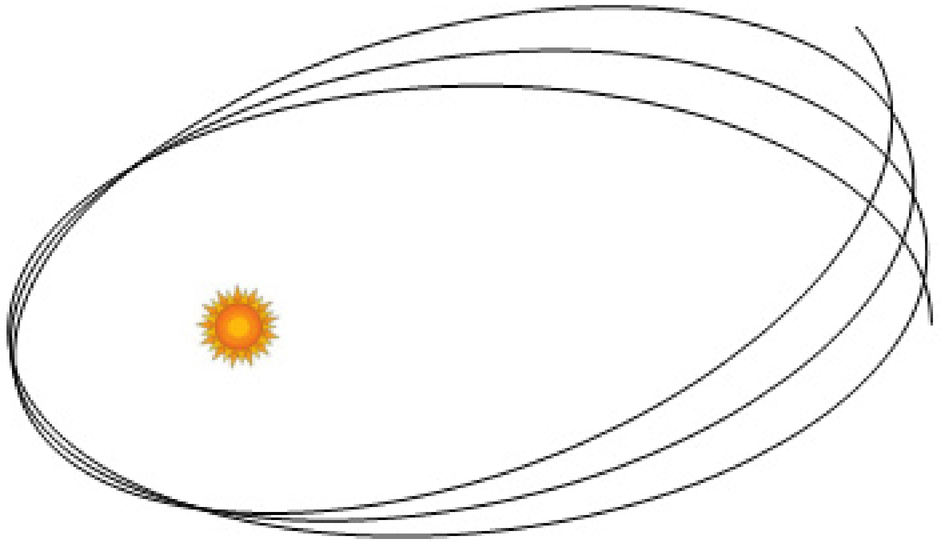
$$G_{ab} = 4\pi \, T_{ab}$$

Some predictions of the General Theory

A. Precession of Mercury

Correction to the Newtonian potential

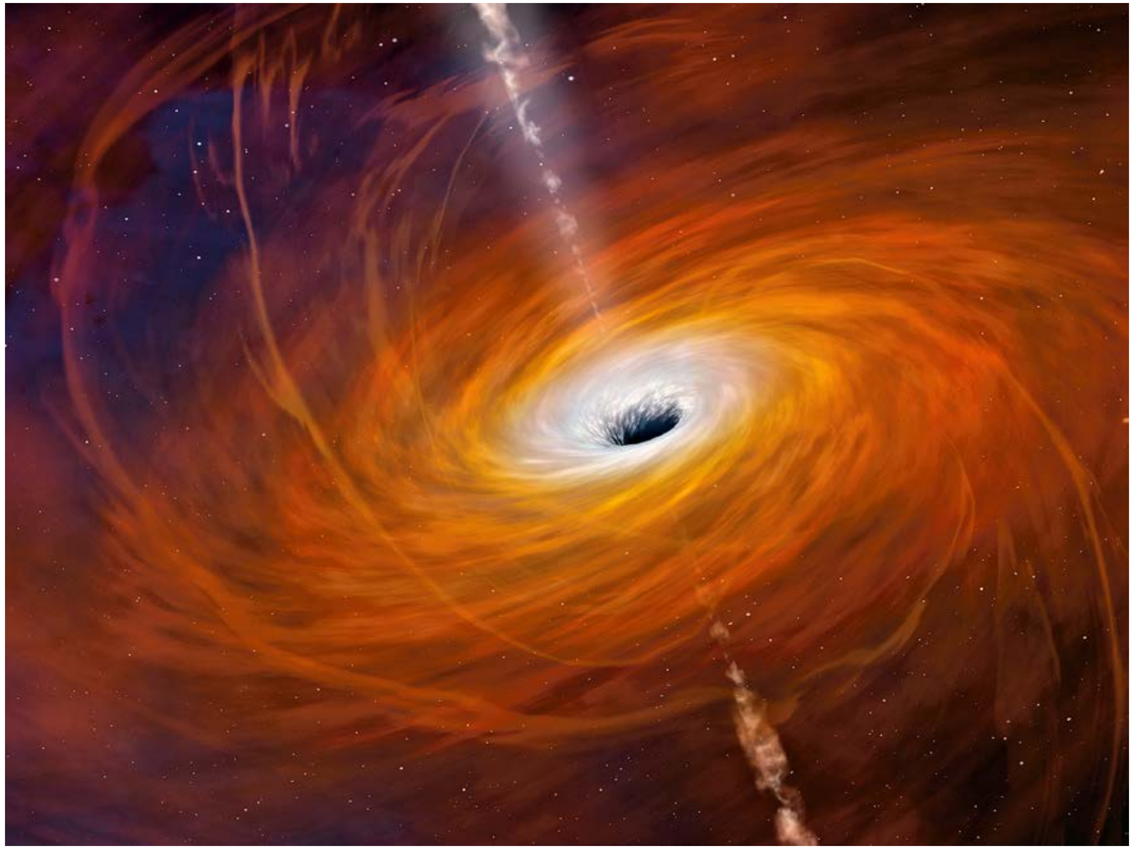
$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GM}{r}$$



Causes the elliptical orbit of Mercury to precess by 23 seconds of arc per orbit.

B. Black Holes

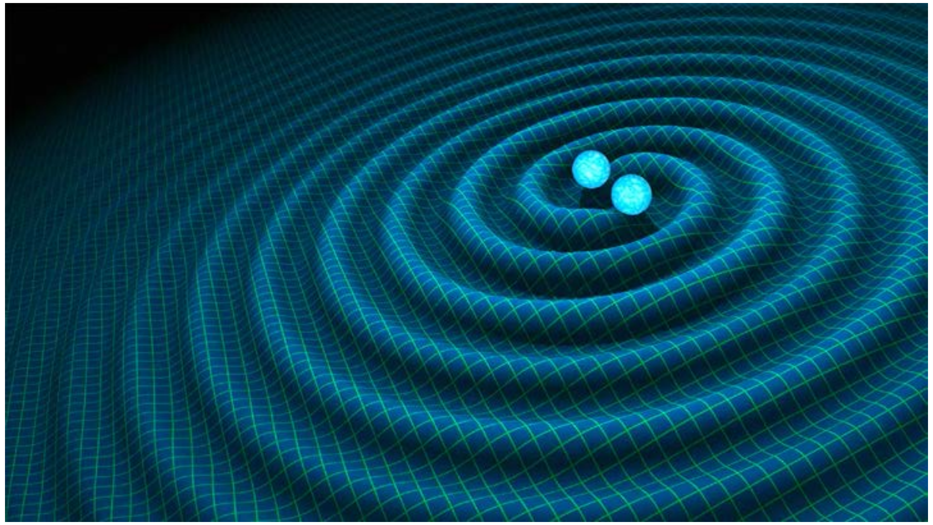
Escape velocity greater than the speed of light



C. Gravitational Waves

Small fluctuations in space-time created by massive cosmic effects in Deep Space

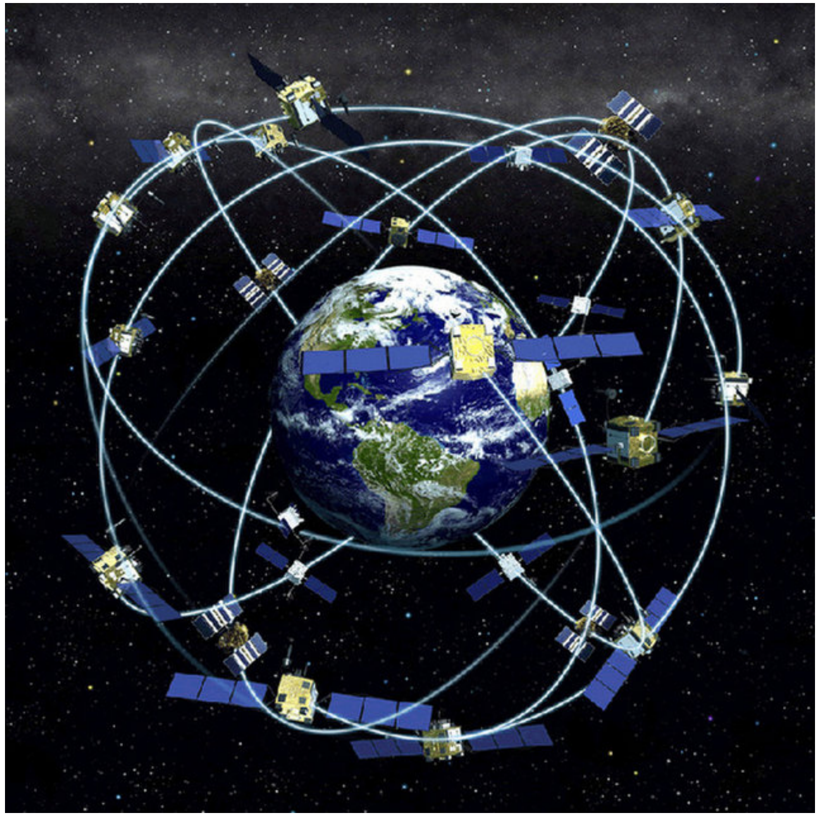
Mathematical prediction 1915 — Discovered experimentally in 2015



Colliding black holes

LIGO Detector

GPS: Bringing Deep Space back to Earth

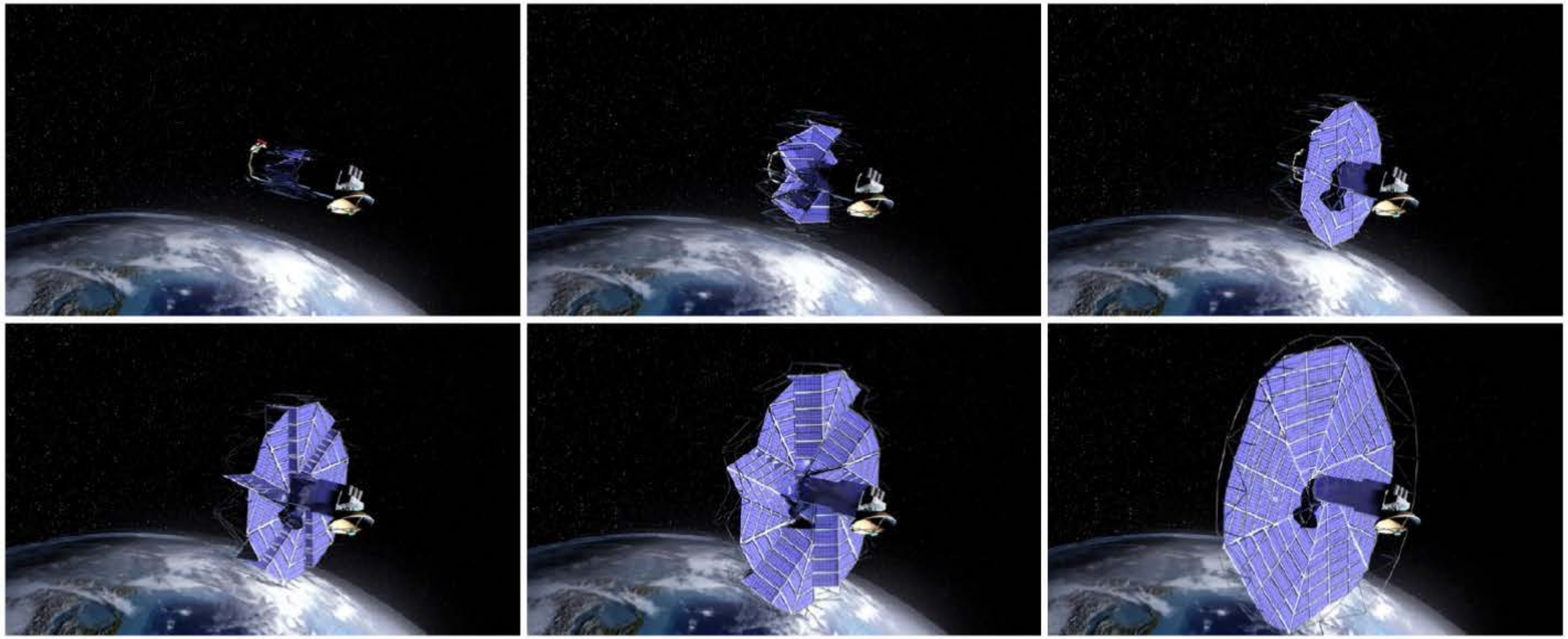


Satellite clocks need to be corrected to allow for the effects of gravity predicted by General Relativity

38 microseconds per day

5. A Bit of Space Origami

Using mathematical origami we can unfold a space craft



Conclusions

Space is big business

Space is out of sight

Mathematics allows us to see further