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SHAPING MODERN MATHEMATICS: POLYNOMIALS AND THEIR ROOTS

PROFESSOR RAYMOND FLOOD

Thank you for coming to the second of my lectures this academic year in my series on shaping modern mathematics. Last time I considered analysis. This week it is going to be algebra.

Let me say first of all what the words in my title mean. A polynomial is an expression in a variable, which we will usually write as x, in which various powers of x can occur multiplied by numbers and added and subtracted together. The highest power of X occurring is called the degree of the polynomial.

Examples of polynomials are
$2X - 6$, a polynomial of degree one or *linear*.
$X^2 - 8X + 15$, a polynomial of degree 2 or *quadratic*.
$X^3 + 18X^2 + 10X - 29$, a polynomial of degree 3 or *cubic*.
$X^4 + X^3 + X^2 + 2X + 1$, a polynomial of degree 4 or *quartic*.
$X^5 + 5X^4 + 10X^3 + 10X^2 + 5X + 1$, a polynomial of degree 5 or *quintic*.
And so on for degree 6, 7, 8 etc as far as you wish to go.

The numbers appearing in the polynomial are called the *coefficients* of the polynomial so for the cubic example the coefficients are 1, 18, 10 and -29. They tell you how many there are of the different powers of x appearing in the polynomial and what is the constant term.

For the quintic example, the coefficients are 1, 5, 10, 10, 5, and 1.
Now for the other term in my lecture title: what the *roots* of a polynomial?

The roots of a polynomial are simply the values of x which make it equal to zero.
With our linear example this means solving the equation
$2X - 6 = 0$ which has solution $X = 3$ so the root of the linear polynomial $2X - 6$ is 3.
For the quadratic polynomial the roots will be obtained by finding those values of X which make
$X^2 - 8X + 15$ equal zero. I can identify two roots.
One is 3 because $3^2 - 8$ times 3 plus 15 is zero.
Another is 5 because $5^2 - 8$ times 5 plus 15 is also zero.
Now that we know what the words in the lecture title mean I can tell what the lecture is going to about!
It is about the hunt to find a formula for the roots of a polynomial, any polynomial, from its coefficients using the four basic arithmetic operations of addition, subtraction, multiplication, division and extracting roots, for example, square roots, cube roots, fourth roots etc.

We can do it for quadratic equations. You might be familiar with, or at least remember meeting, the formula for finding the roots of a quadratic equation, an equation where the highest power of the unknown is two.

$aX^2 + bX + c = 0$

Here the coefficients of the quadratic polynomial are
a, b and c.

There are two roots.

One of the roots is given by $\frac{-b+\sqrt{b^2-4ac}}{2a}$

And the other by $\frac{-b-\sqrt{b^2-4ac}}{2a}$

This is usually written usually $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

In the case of $X^2 - 8X + 15 = 0$ we have
a = 1, b = -8 and c = 15. And so

$x = (8 \pm \sqrt{(-8)^2 - 4.1.15})/2 = (8 \pm \sqrt{64 - 60})/2$
$= (8 \pm 2)/2$

This gives 3 if we take the minus sign or 5 on taking the plus sign.

This formula will work for all quadratic equations, and note, particularly, that it only involves, addition, subtraction, multiplication, division and finding square roots of expressions in the coefficients.

The quest for a similar formula for polynomial equations where the highest power is three, four, five or more led to dramatic changes in how this question was regarded, and indeed was crucial in the creation of modern algebra.

It turns out that there is a formula for cubics and quartics but there is none for quintics and higher degree polynomials! This is an amazing result with major consequences.

Let me give an overview of the rest of the lecture.

I will briefly show that finding the solution of what we call quadratic equations goes back to the Mesopotamians. Islamic mathematicians looked at equations geometrically and made the first serious attempt on some cubic equations. Then we come to one of the most notorious and exciting stories in the history of mathematics when in 16th century Italy formulae were discovered for the cubic and for the quartic, i.e. polynomial equations of degree three and four.

After this, mathematicians tried to find formulae for polynomials of degree five or higher, without success, because it turned out to be impossible.

This result is a nineteenth century story and I want to pick out three themes:
Polynomial equations do have roots! If we are going to seek a formula for the roots of a polynomial equation it would be reassuring to know that it has a solution and it was the great Carl Friedrich Gauss who showed that they do. This is called the Fundamental theorem of Algebra. In my last lecture we talked about the Fundamental Theorem of the Calculus and in my lecture after Christmas I will talk about the Fundamental theorem of Arithmetic. Today I talk about the Fundamental Theorem of Algebra.

Then we look at the work of Abel and Galois, two mathematicians who died tragically young. They showed that it was impossible to find a general formula for solving polynomial equations of degree greater than four, i.e. quintics and higher degree polynomial equations.

Complex or imaginary numbers arise naturally in solving polynomial equations and I want to show you how to remove the mystery that is sometimes associated with them.

Let us start with Mesopotamian (or Babylonian) mathematics. It developed over some 3000 years and over a wide region.

The word *Mesopotamian* comes from the Greek for 'between the rivers'. It refers to the area between the rivers Tigris and Euphrates in modern-day Iraq.

Using a wedge-shaped stylus, the Mesopotamians imprinted their symbols into moist clay – this is called *cuneiform writing* – and the tablet was then left to harden in the sun. Many thousands of mathematical clay tablets have survived.

We write numbers in the decimal place-value system, based on 10, with separate columns for units, tens, hundreds, etc., as we move from right to left. Each place has value ten times the next.

The Mesopotamians also used a place-value system, but it was a 'sexagesimal' system, based on 60: each place has value sixty times the next. Remnants of their sexagesimal system survive in our measurements of time (60 seconds in a minute, 60 minutes in an hour) and of angles.

There were essentially three types of mathematical clay tablet. Some of them list tables of numbers for use in calculations and are called *table tablets.*

On the slide is a drawing of a table tablet showing the nine times table.

Other clay tablets, called *problem tablets,* posed and solved mathematical problems. A third type may be described as *rough work,* created by students while learning.

The problem I want to show you dates from the Old Babylonian period from between 1800 and 1600 BC and is from a clay tablet in the British Museum. The catalogue number is BM 13901. It gives a series of steps or recipe for solving a particular problem of a type that we would now call a quadratic equation.

It does this using essentially the method that I showed you, but in the context of a particular problem.
In our notation, a problem on this tablet gives an algorithm for solving
$X^2 + 1\,X = \frac{3}{4}$
Similar algorithmic approaches can be found in ancient Egyptian texts around the same time.

An important source for these problems is the Rhind papyrus dating from 1650 BC and believed to be copied from a text that is 150 years older. It is also in the British Museum.

Unlike the Egyptians and the Mesopotamians, the Greeks were more concerned with proving things, particularly in arithmetic and geometry, than with such algebraic pursuits as solving equations.

But there was a strand of Greek mathematics that was not constrained by geometry and that was in the work of Diophantus who has been called the father of algebra.

He probably lived in the 3th century AD. We know little about his life. His main contributions to mathematics were the 13 books that comprise his Arithmetica, not all of which survive. Unlike the geometrical writings of most Greek mathematicians this was a collection of algebraic problems that were posed and solved. Diophantus was also the first mathematician to devise and employ algebraic symbols.

Diophantus did not present general methods for solving his problems, but often chose a particular example and found the result in that case alone. The following solution is adapted from the Arithmetica.

To find two numbers such that their sum and product are given numbers.
Given sum 20, given product 96.
Let 2x be the difference of the numbers.
Therefore the numbers are 10 + x, 10 - x.
[Note that 10 is half of the sum, 20.]
Hence, $100 - x^2 = 96$.
So x = 2, and the numbers are 12 and 8.

When Diophantus's work was introduced into the Latin West in the 16th Century it was to be very influential on mathematicians who were still thinking in terms of geometry with its attendant constraints.

Crucial to this transmission of Diophantus's work and indeed that of other Greek mathematicians was the work of Islamic mathematicians, translators and commentators who worked over a period of 700 years from the eight to the fifteenth century. They made significant contributions to algebra and geometry. In particular they studied quadratic and cubic equations. I want to draw your attention to the work of two Islamic mathematicians:

One of the earliest scholars at the House of Wisdom – established in the early 9th century in Bagdad by Caliph Harun al-Rashid and his son - was the Persian scholar Muhammad ibn-Musa (al-)Khwarizmi. The author of two celebrated astronomical star tables and an influential treatise on the astrolabe, he is remembered by mathematicians primarily for his books on arithmetic and algebra. His Arithmetic was important for introducing the Indian number system to the Islamic world and later helping to spread the decimal counting system throughout Christian Europe. Indeed, his Arabic name, transmuted into 'algorism', was later used in Europe to mean arithmetic, and we still use the word algorithm to refer to a step-by-step procedure for solving problems.

The title of al-Khwarizmi's algebra book is Kitab al-jabr wal-muqabala (The Compendious Book on Calculation by Completion [al-jabr] and Reduction [al-muqabala]). This book title is the origin of our word 'algebra': the term 'al-jabr' refers to the operation of transposing a term from one side of an algebraic equation to the other. Al-Khwarizmi's Algebra commences with a lengthy account of how to solve linear equations (with numbers and terms involving x) and quadratic equations (so involving x^2). Since negative numbers were still not considered meaningful, he split the equations into the following six types, given here with their modern equivalents (where a, b and c are positive constants):

roots equal to numbers ($ax = b$)
squares equal to numbers ($ax^2 = b$)
squares equal to roots ($ax^2 = bx$)
squares and roots equal to numbers ($ax^2 + bx = c$)
squares and numbers equal to roots ($ax^2 + c = bx$)
roots and numbers equal to squares ($bx + c = ax^2$)
He then proceeded to solve instances of each type, such as $x^2 + 10x = 39$ using a geometrical form of 'completing the square'.

We can use essentially this approach to obtain the formula for the solution of a quadratic equation seen earlier.

Recall the slide we had earlier stating that that the solution of
$a x^2 + b x + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is because we can "complete the square" to show that
$a x^2 + b x + c$ can be written as $a(x + \frac{b}{2a})^2 - a(\frac{b}{2a})^2 + c$

And therefore
$a(x + \frac{b}{2a})^2 = a(\frac{b}{2a})^2 - c$

giving, on dividing by a
$(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$

And taking square roots gives
$(x + \frac{b}{2a}) = \pm\sqrt{(\frac{b}{2a})^2 - \frac{c}{a}}$

This can be rearranged to give

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Persian poet Omar Khayyam (1048–1131), remembered mainly for his collection of poems called the Rubaiyat, was also a mathematician who wrote on algebra, geometry and the calendar. In his writings on algebra Omar Khayyam presented the first systematic classification of cubic equations (those involving x^3), similar to al-Khwarizmi's classification of linear and quadratic equations. He also presented a geometrical method for solving several types of cubic equations;

A solid cube plus edges equal to a number
In modern notation, $x^3 + cx = d$.
An example is $x^3 + 16x = 40$. His method was to draw a particular semicircle
$x^2 + y^2 = (d/c)x$. *in the example it is* $x^2 + y^2 = (40/16)x$
and a particular parabola
$x^2 = \sqrt{16}\, y$
and find the point where they intersect: a solution x is as marked. In fact it is x = 2

Al-Khwarizmi and Khayaam's work was not known in the West until much later. The same was true of the work of Indian mathematicians who had also solved quadratic equations.

I now want to move the scene to Western Europe.

Johann Gutenberg's invention of the printing press around 1440 revolutionized mathematics, enabling classical mathematical works to be accessible for the first time. Previously, scholarly works, such as the classical texts of Euclid and Archimedes, had been available only to scholars in manuscript form, but the printed versions made these works much more widely available, just as the internet does today.

Of particular importance to our story was an important and influential vernacular text, first published in 1494, by Luca Pacioli (1447–1517), a mathematics teacher and Franciscan friar. It was the *Summa de arithmetica, geometrica, proportioni et proportionalita* (Summary of Arithmetic, Geometry, Proportion and Proportionality). This is a 600-page compilation of the mathematics known at the time, written in Italian for his students. It is now remembered mainly for including the first published account of double-entry bookkeeping – with the result that Pacioli is sometimes called "the Father of Accounting". However it also highlighted a key problem.

Could a formula to solve cubic equations be found and could it be justified geometrically in the same way that there was a formula for the quadratic formula with a geometrical justification? We have seen how Omar Khayyam classified cubic equations and solved one by intersecting a semicircle with a parabola. But little further progress was made on solving cubic equations in general, and even around 1500 Pacioli and others were pessimistic as to whether this could be done.

The attempt to solve cubic equations is one of the most celebrated stories in the history of mathematics. It took place in Bologna in the early 16th century, during a period when Italian University academics had little job security. Having to compete annually for their positions, they often had to prove their superiority over their rivals by resorting to public problem solving contests.

In the 1520s, Scipione del Ferro, a Mathematics lecturer at the University of Bologna, found a general method for solving cubic equations of the form *A cube and things equal to numbers* (which we would write as $x^3 + cx = d$), and revealed it to his pupil Antonio Fior.

Another who investigated cubic equations around this time was Niccolo of Brescia (1499/1500–1557), known as Tartaglia (the stammerer) from a bad stammer that he developed after being slashed by a sword across the face when young.

In particular, Tartaglia found a method for solving equations of the form *A cube and squares equal to numbers* (which we would write as $x^3 + bx^2 = d$).

After del Ferro's death in 1526, Fior felt free to exploit his secret, and challenged Tartaglia to a cubic-solving contest. Fior presented him with thirty cubic equations of the first form, giving him a month to solve them. Tartaglia in turn presented Fior with thirty cubic equations of the second form.

Fior lost the contest. He was not a good enough mathematician to solve Tartaglia's type of problem, while Tartaglia, during a sleepless night ten days before the contest, managed to discover a method for solving all of Fior's problems. The essence of Tartaglia's method was to reduce the cubics he had to solve to solving a quadratic which, of course, he knew how to do.

Meanwhile in Milan, Gerolamo Cardano (1501–1576) was writing extensively about a range of topics, from physics and medicine to algebra and probability (especially its applications to gambling). On hearing about the contest, Cardano determined to prise Tartaglia's method out of him.

This he did one evening in 1539, after promising Tartaglia an introduction to the Spanish Governor of the city. Tartaglia hoped that the Governor would fund his researches, and in turn extracted from Cardano the following solemn oath not to reveal his method of solution:

I swear to you, by God's holy Gospels,
and as a true man of honour, not only never
to publish your discoveries, if you teach me
them, but I also promise you, and I pledge
my faith as a true Christian, to note them
down in code, so that after my death, no-one
will be able to understand them.

However, in 1542 Cardano learned that the original discovery of Tartaglia's method had been due to del Ferro, and he felt free to break his oath.

Meanwhile, his brilliant colleague, Ludovico Ferrari, had found a similar general method for solving quartic equations (involving terms in x^4). He reduced the problem of solving the quartic to that of solving cubics which they by now knew how to do.

In 1545, Cardano published *Ars Magna* (The Great Art), containing the methods for solving cubics and quartics and giving credit to Tartaglia. The *Ars Magna* became one of the most important algebra books of all time, but Tartaglia was outraged by Cardano's behaviour and spent the rest of his life writing him vitriolic letters. Thus, after a struggle lasting many centuries, cubic equations had at last been solved, together with quartic equations.

In the Ars Magna Cardano presented geometric justifications for solving the various cases of the cubic, essentially completing the cube where al-Khwarizmi had completed the square.

But he did not geometrically justify Ferrari's algorithms for solving the quartic remarking

"All those matters up to and including the cubic, are fully demonstrated, but the others which we will add, either by necessity or out of curiosity, we do not go beyond barely setting out."

Here we find algebra breaking out of the geometric cage in which it had been locked.

This is a good point to pause and sum up where we have reached. We have formulas for solving linear, quadratic, cubic and quartic.

But I have been shielding you from something. These formulae could give rise to the square root of -1 which appears nonsensical and mysterious as no ordinary number can be the square root of -1. This is because the square of any number is never negative. The square root of -1 came to be called complex or imaginary and for many centuries they were regarded with suspicion.

The quadratic on the screen arose from a numerical problem that Cardano tried to solve. It was:
Divide 10 into two parts whose product is 40.

Cardano could see no meaning to these solutions but observed
Nevertheless we will operate, putting aside
the mental tortures involved,
and found that everything worked out correctly:
In view of these *mental tortures*, Cardano was led to complain that:
So progresses arithmetic subtlety, the end of
which is as refined as it is useless.

$(5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10$

$(5 + \sqrt{-15}) \times (5 - \sqrt{-15}) = 40$

Euler, in the eighteenth century, who worked with this type of number a great deal, also said:
Of such numbers we may truly assert that they
are neither nothing, nor greater than nothing,
nor less than nothing, which necessarily
constitutes them imaginary or impossible.

Even in the early 19th century there was still a great deal of unhappiness about complex numbers, and about so-called 'imaginary' numbers that don't seem to exist. For example,
Augustus De Morgan, Professor of Mathematics at University College, London, declared that:
We have shown the symbol $\sqrt{-1}$ *to be void of meaning, or rather self-contradictory and absurd.*

It was William Rowan Hamilton who finally provided an explanation of complex numbers that was generally accepted. He proposed that the complex number should be defined as a pair (a, b) of real numbers. We combine such pairs (a, b) and (c, d) by using the following rules:

Addition:
$(a, b) + (c, d) = (a + c, b + d)$;
this corresponds to the equation

Multiplication:
$(a, b) \times (c, d) = (ac - bd, ad + bc)$;

The pair $(a, 0)$ then corresponds to the real number a, the pair $(0, 1)$ corresponds to the
imaginary number $i = \sqrt{-1}$, and we have the equation
$(0, 1) \times (0, 1) = (-1, 0)$,
which corresponds to the equation $i \times i = -1$.

In this representation, called the complex plane, two axes are drawn at right angles – the real axis and the imaginary axis – and the complex number $a + b\sqrt{-1}$ is represented by the point at a distance a in the direction of the real axis, which is horizontal, and at height b in the direction of the imaginary axis, which is vertical.

So far I have been talking about the quest to find a formula for the roots of a polynomial. But this overlooks one crucial question. How do we know that every polynomial has roots?

We can establish this quite easily in the case of odd degree polynomials.

In this example the core thing to notice is that as x becomes big and positive the highest degree term x^5 will swamp all the other terms in the sense of being much bigger than them and positive. So the graph will go upwards on the right hand side.

On the other hand if x is large and negative then x^5 is even larger and also negative and again swamps the other terms so the graph will go downwards on the left hand side.

Hence the graph is below the horizontal axis on the left and above it on the right and so must cross the horizontal axis in between. But where it crosses the horizontal axis is a root of the polynomial.

This argument does not help very much in finding a root but it does show that a root exists.

However this argument need not work for even degree polynomials as we can see in this example. This is because as x gets large and positive then the graph goes up as before. But as x gets large and negative the graph also goes up because any negative number to the power of 6 (or any even power) is positive. And, in between, the graph need not cross the axis as shown here.

However if we allow x to be a complex number we can make a similar technique work. But we have to think in the complex plane. We now think of the geometrical representation of complex numbers and compare the values the polynomial takes when x, now a complex number, goes around a small circle to the values it takes when it goes around a large circle. Once again you can show that a root has to exist – only this time it is a complex number.

Let me now state the Fundamental Theorem of Algebra.

For example
$X^4 - 1 = (X^2 + 1)(X - 1)(X + 1)$

Each linear factor gives a root e.g. X – 1 has root 1 and X + 1 has root -1.
Each quadratic factor has two roots e.g. $(X^2 + 1)$ has roots i and –i. we know how to do this as we have seen the formula for a quadratic often enough.

This tells us that any polynomial equation will have solutions although they could and indeed often will be complex numbers.

Now back to the main story:

Furthermore we have formulae for finding them in the case of linear, quadratic cubic and quartic equations.

The next question was
Can one solve equations involving x^5

The problem was to find a formula for a solution of the quintic equation
$a x^5 + b x^4 + c x^3 + d x^2 + e x + f = 0$
in terms of the coefficients a, b, c, d, e, f, using only the arithmetical operations of +, -, × and ÷ as well as taking roots, $\sqrt[k]{}$, for any k.

It does not seem an unreasonable request to be able to do it for quintics. After all, the equation is made up in terms of powers of x and adding, subtracting and multiplying so it seems possible to find a solution by undoing these operations that is taking roots, subtracting, adding, multiplying and dividing. And we are able to do this for linear equations, quadratic equations, cubic equations and quartic equations. And it remained open until the 19th century when it was shown impossible by Niels Henrik Abel and further developed by Evariste Galois.

The stories of Abel (1802–1829) and Évariste Galois are tragic (1811–1832) and depressingly similar. Both found it difficult to get their results accepted, and although both made major advances in the theory of equations — Abel proved that no general solution can exist for polynomial equations of degree 5 or more, while Galois determined when such equations *can* be solved — both died young, Abel from tuberculosis and Galois after being wounded in a duel.

Growing up in Norway, Abel was desperate to study in the main centres of mathematical life in France and Germany, and was eventually able to obtain financial support that enabled him to spend time in Paris and Berlin. In Germany he met Leopold Crelle and published many papers in the early issues of Crelle's new journal, thereby helping it to become the leading German mathematical periodical of the 19th century; among these papers was the one that contained his proof of the impossibility for solving the general equation of degree 5 or more.

He also obtained fundamental results on other topics. The story of Abel's attempts to be recognized by the mathematical community, and of his lack of success in securing an academic post, is a sorry one. He returned to Norway where he contracted tuberculosis and died at the early age of 26. Two days later, a letter arrived at his home offering him a prestigious professorship in Berlin.

The work of Abel and earlier mathematicians on the unsolvability of the general quintic equation was developed by the brilliant Evariste Galois.

Galois's teenage years were traumatic. He failed his entrance examination for the Ecole Polytechnique. A manuscript that he sent to the French Academy of Sciences was mislaid, another was rejected for being obscure, and his father committed suicide. A republican firebrand who became involved with political activities following the July Revolution of 1830, Galois threatened the life of King Louis-Philippe, but was acquitted. A month later he was discovered carrying weapons and wearing the uniform of the banned artillery guard, whereupon he was thrown into jail.

Galois spent the night before his duel frantically scribbling a letter to his friend August Chevalier, summarizing his results and requesting Chevalier to show them to Gauss and Jacobi. But it was to be several years before anyone appreciated what they meant, and what a genius the world had lost.

He considered a more refined question:

Given a particular equation with numerical coefficients is there a solution in terms of radicals?

A solution in terms of radicals means, in terms of the coefficients a, b, c, d, e, f, using only the arithmetical operations of +, -, × and ÷ as well as taking roots, $\sqrt[k]{}$, for any k.

Sometimes the answer is yes: $X^5 + 5X^4 + 10X^3 + 10X^2 + 5X - 1 = 0$
and
Sometimes the answer is **no:** $X^5 + 10X^2 - 2 = 0$

How do we tell them apart?

Galois determined criteria (in terms of an object now called the *Galois group*) for deciding *which* polynomial equations can be solved by arithmetical operations and the taking of roots. His work ultimately led to whole new areas of algebra, now known as *group theory* and *Galois theory.*

Crucial to the work of Abel and Galois was the notion of permutations and the structure that they could have. Following their work powerful techniques in algebra were developed.

Algebra changed dramatically throughout the 19th century. In 1800 the subject was about solving equations, but by 1900 it had become the study of mathematical structures — sets of elements that are combined according to specified rules, called axioms.

Also the mystique concerning complex numbers was at last removed by William Rowan Hamilton, who defined them as pairs of real numbers with certain operations. Other algebraic structures were discovered: Hamilton introduced the algebra of quaternions, George Boole created an algebra for use in logic and probability, and Cayley studied the algebra of rectangular arrays of symbols, called matrices.

Some of these developments in modern algebra will be in my lecture series next academic year.
Thank you and I leave you with a date for your diary:

A Christmas treat on Tuesday 11th December: when I will speak on developments in geometry.